Gravitational Lensing - I

- **Possibility of Light Rays Being Deflected by Mass Discussed By Newton, Soldner and Laplace**
  - **Typical Derivation is to Use Hyperbolic Orbital Asymptotes and Compute the Deflection**
  - **Classical Result:** $\Delta\theta = \frac{2GM}{c^2 R}$
- **G.R. Predicts the Deflection is Actually Twice the Classical Value Due to Space Time Curvature**
- **Various Approaches to the Derivation but Simplest (Schneider et al. 1994)**
  - **Begin by Considering a Perturbed Minkowski Metric**
  - **Calculate the New Line-element and Note That Light Follows Null Geodesic (ds = 0)**
  - **Then Assigning an Effective Index of Refraction to Space-time:**

    $n = 1 - \frac{2}{c^2}\Phi$ **(see Appendix)**
  - **This is Analogous to Deriving the Deflection of Light Through a Prism via Fermat's Principle, but for GR <u>n is Not a Constant</u>. Specifically:**

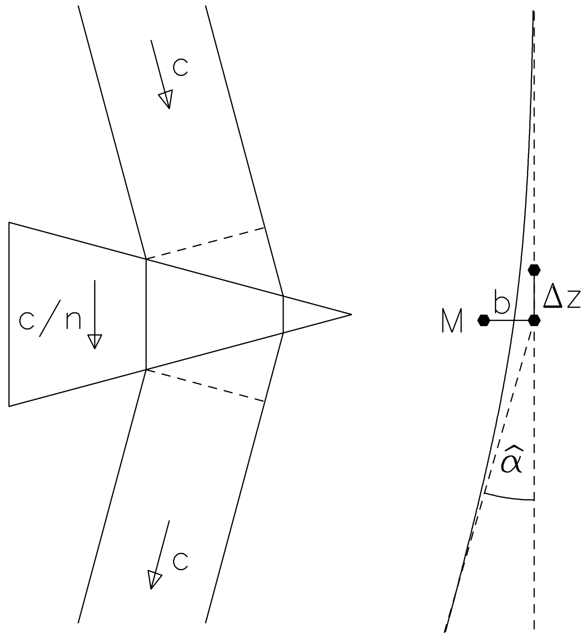
$\alpha = -\int \nabla_\perp n dl = \frac{2}{c^2}\int \nabla_\perp \Phi dl$, **with** $\Phi(b,z) = -\frac{GM}{(b^2+z^2)^{1/2}}$ **from which:**

$\nabla_\perp \Phi(b,z) = \frac{GMb}{(b^2+z^2)^{3/2}}$ **Giving:**

$\alpha = \frac{2}{c^2}\int \nabla_\perp \Phi dl = \frac{4GM}{bc^2} \rightarrow \alpha(\xi) = \frac{4GM(\xi)}{\xi c^2}$

Note: this is twice the classical result given above. For extended mass distributions we replace the mass with the projected mass surface density:

$$M(\xi) = 2\pi \int_0^{\xi} \Sigma(\xi')\xi' d\xi'$$



Left: Light rays traveling through a prism are bent (v < c) so that the travel time for both is the same. Angle depends on pathlength difference if n is constant. Right: Angular deflection of a light ray (α) passing a mass M with impact parameter b. Deflection must be integrated along dz but we can approximate it over Δz.

Lensing Geometry & Lens Equation

A light ray from source S is deflected by the angle $\hat{\alpha}$ to an observer at O. For a deflecting mass on the optical axis and a source displaced by an angle β the image (I) is found at angle θ. The various distances will be cosmological angular size distances. If we call the reduced deflection angle α:

$$\alpha = \frac{D_{ds}}{D_s}\hat{\alpha}$$

Since $\theta D_s = \beta D_s + \hat{\alpha} D_{ds}$ we see that:

$$\beta = \theta - \hat{\alpha}$$

Thus: $\beta(\theta) = \theta - \frac{D_{ds}}{D_s D_d}\frac{4GM(\theta)}{c^2\theta}$

For a source on the axis of the deflecting mass β = 0 the image is a ring (Einstein ring):

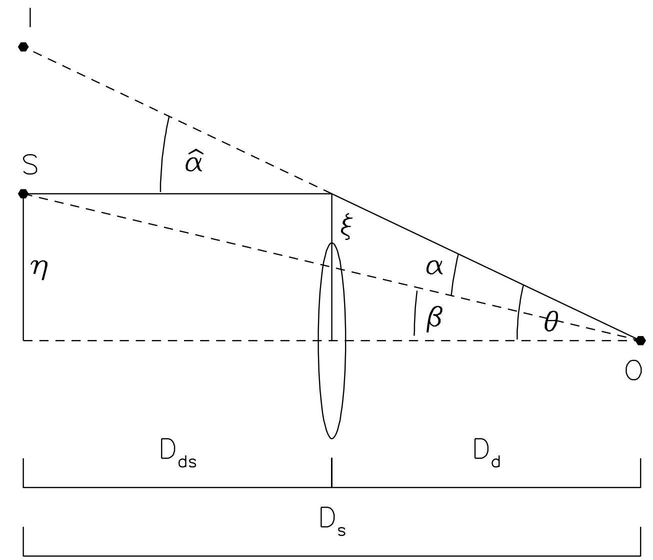
$$\theta_E = \left[\frac{4GM(\theta_E)}{c^2}\frac{D_{ds}}{D_d D_s}\right]^{1/2}$$

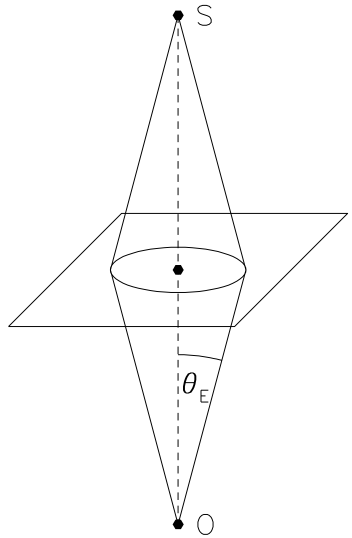
For a point mass we can rewrite the lens equation as:

$\beta = \theta - \frac{\theta_E^2}{\theta}$ **and solving for θ we find two solutions:**

$$\theta_{\pm} = \frac{1}{2}\left(\beta \pm \sqrt{\beta^2 + 4\theta_E^2}\right)$$

That is, a displaced source s is imaged twice by a point mass, one inside the Einstein Radius and one outside.





Lensing Magnification

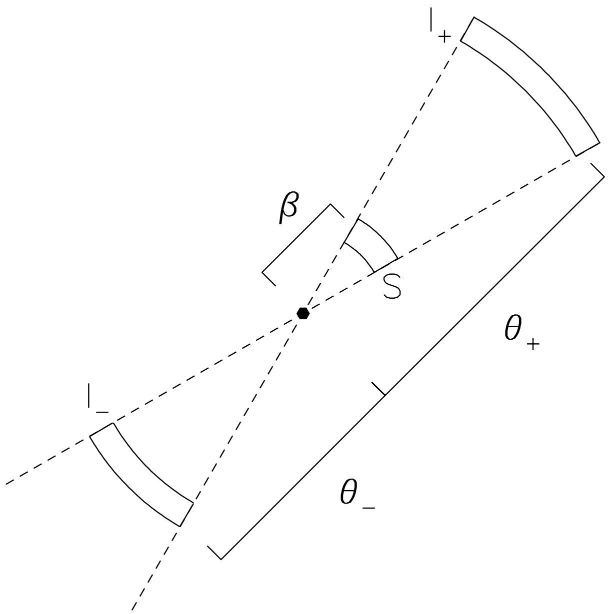
Gravitational lensing changes the apparent solid angle of the source, but surface brightness is conserved so we can define the brightness magnification as:

$\boldsymbol{magnification} = \frac{\boldsymbol{image\ area}}{\boldsymbol{source\ area}}$ or $\boldsymbol{\mu} = \frac{\boldsymbol{\theta}}{\boldsymbol{\beta}}\frac{\boldsymbol{d\theta}}{\boldsymbol{d\beta}}$, and so for a point mass we have:

$$\boldsymbol{\mu} \pm = \left[\mathbf{1} - \left(\frac{\boldsymbol{\theta_E}}{\boldsymbol{\theta} \pm}\right)^{\mathbf{4}}\right]^{-\mathbf{1}} = \frac{\boldsymbol{u}^{\mathbf{2}} + \mathbf{2}}{\mathbf{2}\boldsymbol{u}\sqrt{\boldsymbol{u}^{\mathbf{2}} + \mathbf{4}}} \pm \frac{\mathbf{1}}{\mathbf{2}}$$

where u is the angular separation of the source from the mass in units of the Einstein radius: $\boldsymbol{u} = \boldsymbol{\beta}\boldsymbol{\theta}_{\boldsymbol{E}}^{-\mathbf{1}}$. Negative magnifications indicate a parity inversion (flip) of the images as shown in the figure. If the two images are unresolved, as in microlensing, the net magnification is a useful measure:

$\boldsymbol{u} = |\boldsymbol{u}_+| + |\boldsymbol{u}_-| = \frac{\boldsymbol{u}^{\mathbf{2}}+\mathbf{2}}{\boldsymbol{u}\sqrt{\boldsymbol{u}^{\mathbf{2}}+\mathbf{4}}}$ and so when a source is located at the Einstein ring for the point mass the net magnification is 1.34. Smaller impact parameters result in greater magnifications and larger in less.



A source S displaced from a point mass by angle β with images I_+ and I_- found at positions θ_+ and θ_-.

Lensing by Galaxies and Clusters

- **Lensing from Extended Mass Distributions Requires Knowing or Modeling the Mass Distribution. Consider the Singular Isothermal Sphere:**

$$\rho(r) = \frac{\sigma_v^2}{2\pi G r^2}$$

Integrating along the line-of-sight (Abel integral) give the projected mass density:

$$\Sigma(\xi) = \frac{\sigma_v^2}{2G\xi}$$

and the deflection angle is :

$$\hat{\alpha} = 4\pi\frac{\sigma_v^2}{c^2}$$

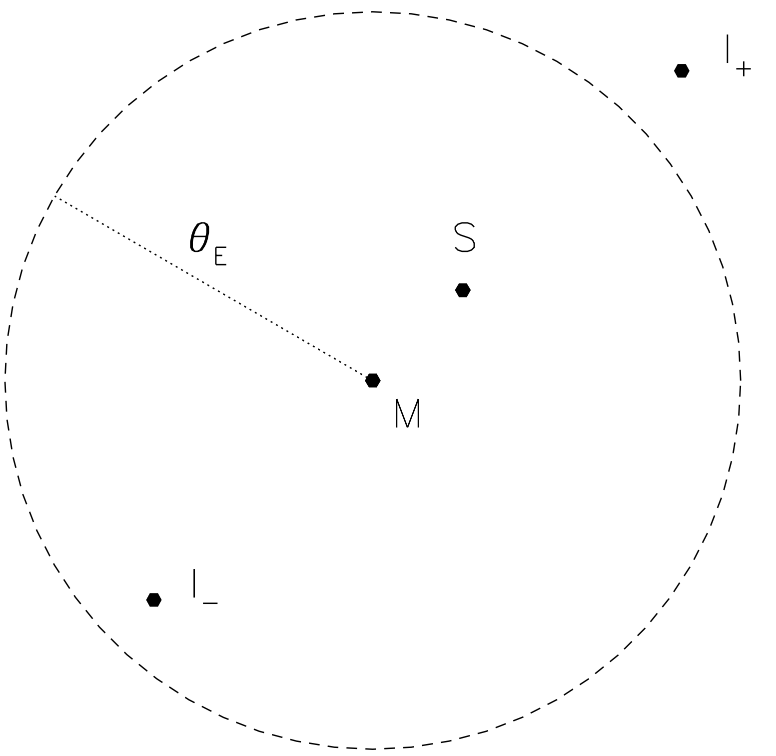
which is independent of impact parameter ξ. The Einstein radius is:

$\theta_E = \frac{\sigma_v^2}{c^2}\frac{D_{ds}}{D_s} = \hat{\alpha}\frac{D_{ds}}{D_s} = \alpha$ **and the lens equation has two solutions:**

$$\theta \pm = \beta \pm \theta_E$$

With the two images located along a line containing the axis of the mass and the projected source position. The magnification is:

$$\mu_\pm = \frac{\theta_\pm}{\beta} = 1 \pm \frac{\theta_E}{\beta}$$



Effective Lensing Potential

- **Next Let's Define an Effective Lensing Potential**

First Define the <u>Critical Mass Density</u>:

$$\Sigma_{CR} = \frac{c^2}{4\pi G}\frac{D_s}{D_d D_{ds}}$$

Note: for the case of Σ_{CR} = constant, $\alpha(\theta) = \theta$ and the lens actually has a focal length. Think of it as constant index of refraction. Let's now rescale the lensing potential for distances as an <u>effective lensing potential</u>:

$$\psi(\theta) = \frac{D_{ds}}{D_d D_s}\frac{2}{c^2}\int \Phi(D_d\theta, z)dz$$

The derivatives have interesting properties. Gradient:

$$\nabla_\theta \psi = D_d \nabla_\xi \psi = \frac{2}{c^2}\frac{D_{ds}}{D_s}\int \nabla_\perp \Phi dz = \alpha$$

and for the Laplacian:

$$\nabla_\theta^2 \psi = \frac{2}{c^2}\frac{D_d D_{ds}}{D_s}\int \nabla_\xi^2\, \Phi dz = \frac{2}{c^2}\frac{D_d D_{ds}}{D_s}\cdot 4\pi G\Sigma$$

$$= 2\frac{\Sigma(\theta)}{\Sigma_{CR}} \equiv 2\kappa(\theta)$$

Where the surface mass density scaled with its critical value is κ(θ) and is called the <u>convergence</u>.

Effective Lensing Potential: Convergence & Shear

The effective potential can be written in terms of the convergence:

$\psi(\theta) = \frac{1}{\pi}\int \kappa(\theta')\ln|\theta - \theta'|d^2\theta'$

Since the deflection angle is the gradient of ψ:

$\alpha(\theta) = \nabla\psi = \frac{1}{\pi}\int \kappa(\theta')\frac{\theta-\theta'}{|\theta-\theta'|^2}d^2\theta'$. **These formalisms allow us to then write the lens mapping equation as a Jacobian matrix A. This becomes important when we consider non-axisymmetric lensing masses:**

$$A \equiv \frac{\partial\beta}{\partial\theta} = \left(\delta_{ij} - \frac{\partial\alpha_i(\theta)}{\partial\theta_j}\right) = \left(\delta_{ij} - \frac{\partial^2\psi(\theta)}{\partial\theta_i\partial\theta_j}\right) = M^{-1}$$

A is evidently the inverse of the magnification tensor and it describes the mapping of a solid angle of the source into a solid angle of the image. The magnification tensor thus describes the distortion of the images:

$$\frac{\partial\theta^2}{\partial\beta^2} = detM = \frac{1}{detA}$$

But a matrix of second partial derivatives of ψ (called the Hessian) describes the deviation from an identical mapping (i.e., not just a magnification). If we define $\psi_{ij} \equiv \frac{\partial^2\psi}{\partial\theta_i\partial\theta_j}$ we can write the convergence as:

$$\kappa = \frac{1}{2}(\psi_{11} + \psi_{22}) = \frac{1}{2}tr\psi_{ij}$$

And two linear combinations of ψ_{ij} can be used to define the <u>shear tensor</u>:

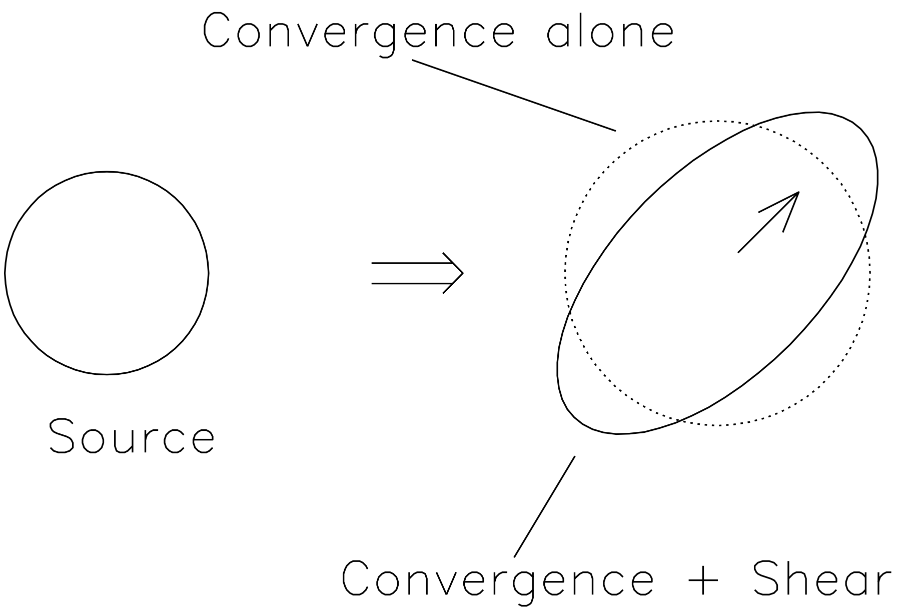
$\gamma_1(\theta) = \frac{1}{2}(\psi_{11} - \psi_{22}) \equiv \gamma(\theta)cos[2\phi(\theta)]$, **and** $\gamma_2(\theta) = \psi_{12} = \psi_{21} \equiv \gamma(\theta)sin[2\phi(\theta)]$

Effective Lensing Potential: Convergence & Shear

The Jacobian matrix now becomes:

$$A = \begin{pmatrix} 1-\kappa-\gamma_1 & -\gamma_2 \\ -\gamma_2 & 1-\kappa+\gamma_1 \end{pmatrix}$$

$$= (1-\kappa)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \gamma \begin{pmatrix} cos2\phi & sin2\phi \\ sin2\phi & -cos2\phi \end{pmatrix}$$



Now we see that the Jacobian (a function of θ) can be thought of as two parts: a convergence (magnification) alone portion and a shear (distortion) that operates on a light cone (solid angle) from the source (see figure). The magnification is the determinant of A as before:

$$\mu = detM = \frac{1}{detA} = \frac{1}{[(1-\kappa)^2 - \gamma^2]}$$

All of this gives us the nomenclature to consider elliptical mass distributions as well as examine other lensing properties, such as time delay.

Time Delays in Gravitational Lensing

- **Conceptually, time delay occurs due to both geometric path length differences and time dilation, a function of the potential depth (see figure).**
  - **Result is a time delay surface that is a function of angular position.**
  - **A given source can produce multiple images if its position is inside the Einstein radius. The time delay for these different ray paths is necessarily different.**
  - **A variable source, e.g., quasar or supernovae will exhibit light curve delays between the different images that correspond to these geometric and dilation effects.**
- **The Hessian of the potential maps the local curvature of the time delay surface:**

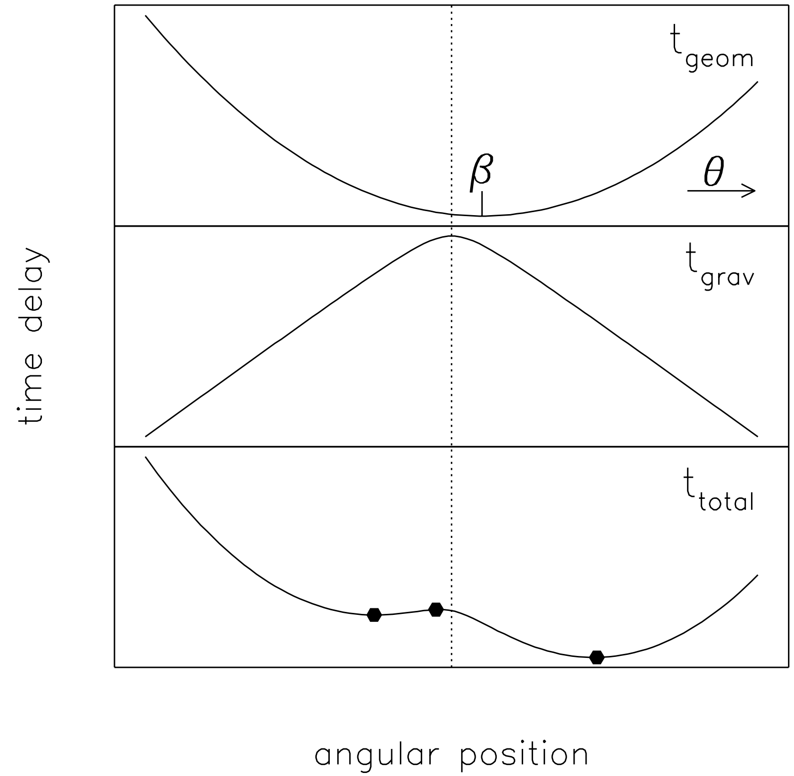
$$\boldsymbol{T = \frac{\partial^2 t(\theta)}{\partial\theta_i \partial\theta_j} \propto \left(\delta_{ij} - \psi_{ij}\right) = A}$$

- **Images can be grouped according to where they are located on the time delay surface:**

  Type I: images located at a minimum of t(θ), *det A* > 0, *tr A* >0, det A, positive magnification,

  Type II: images located at a saddle point, *det A* < 0 (eigenvalues have opposite sign), negative magnification,

  Type III: images located at a maximum of t(θ), > 0, *det A* > 0, *tr* A < 0, both eigenvalues are negative, positive magnification.



Time delays are composed of two parts: a geometric part due to path length differences and a part due to gravitational time dilation.

References

- Gravitational Lenses, Schneider, Ehlers & Falco 1992, (Berlin: Springer Verlag)
- Lectures on Gravitational Lensing, Narayan & Bartelmann, in Formation of Structure in the Universe, Dekel & Ostriker 1995, (Cambridge: Cambridge Univ. Press)

Appendix - I

- **Derivation of Effective Index of Refraction of Space-time**

The unperturbed Minkowski metric is:

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

whose line element is: $ds^2 = c^2 dt^2 - (dx)^2$. **A weak mass perturbs this metric such that:**

$$\eta_{\mu\nu} \rightarrow g_{\mu\nu} = \begin{pmatrix} 1 + \frac{2\Phi}{c^2} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\Phi}{c^2}\right) & 0 & 0 \\ 0 & 0 & -\left(1 - \frac{2\Phi}{c^2}\right) & 0 \\ 0 & 0 & 0 & -\left(1 - \frac{2\Phi}{c^2}\right) \end{pmatrix}$$

whose line element becomes: $ds^2 = \left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi}{c^2}\right)(dx)^2$. **Since light rays follow null geodesics** $ds = 0$ **so:**

$$\left(1 + \frac{2\Phi}{c^2}\right) c^2 dt^2 = \left(1 - \frac{2\Phi}{c^2}\right)(dx)^2$$

So that the light speed in the gravitational field is thus:

$$c' = \frac{|dx|}{dt} = c\sqrt{\frac{1 + \frac{2\Phi}{c^2}}{1 - \frac{2\Phi}{c^2}}} \approx c\left(1 + \frac{2\Phi}{c^2}\right)$$

So that we can define a space-time index of refraction:

$$n = \frac{1}{1 + \frac{2\Phi}{c^2}} \approx 1 - \frac{2\Phi}{c^2}$$