Galactic Dynamics

Primary Reference -
Galactic Dynamics - Binney and Tremaine

Background

Recall the Virial Theorm:
Let the kinetic energy be T and the potential energy be W The total energy is thus:

$$E = T + W \qquad T = \tfrac{1}{2} m \langle v^2 \rangle \qquad W = -\frac{Gm^2}{\langle R \rangle}$$

Virial Theorm (Scaler)

$$2T + W = 0$$ so:

$$E = -T = W/2$$ (half potential energy is lost)

Gravitational Collapse + Virialization means:

$$W' = 2W \Rightarrow R' = \tfrac{1}{2} R$$

Note that gravity has no natural scale:

$$F \propto r^{-2}$$

But relativistic effects occur on scales of a few times the Einstein radius:

$$R_E = \frac{2GM}{c^2}$$ (not important here)

$\Rightarrow$ Scale of stellar systems set by other processes such as gas dynamics.

Potential Theory (BT ch. 2)

Recall that

$$\underset{\text{(vector)}}{F_c} = -\nabla \underset{\text{(scaler)}}{\Phi}$$

Poisson's Equation (relation between potential and density)

$$\nabla^2 \Phi = 4\pi G \rho$$

For a point mass (outside spherical):

$$F(r) = -\frac{d\Phi}{dr}\hat{e}_r = -\frac{GM(r)}{r^2}\hat{e}_r$$

where $M(r) = 4\pi \int_0^r \rho(r') r'^2 dr'$

But since $V_c^2/r = GM(r)/r^2$ then:

$$V_c^2 = r\frac{d\Phi}{dr} \text{ so}$$

$$V_c = \sqrt{\frac{GM(r)}{r}}$$ But energy is conserved

so: $\frac{1}{2}V_e^2(r) + \Phi(r) = 0$ so:

$$V_e = \sqrt{2|\Phi(r)|}$$

potential energy is negative

So for a point mass:

$$\Phi(r) = -\frac{GM}{r}, \quad V_c(r) = \sqrt{\frac{GM}{r}}, \quad V_e(r) = \sqrt{\frac{2GM}{r}}$$

(Keplerian)

and for a homogeneous sphere:

$$M(r) = \tfrac{4}{3}\pi r^3 \rho \quad \text{and} \quad V_c = \sqrt{\frac{4\pi G\rho}{3}}\, r$$

For the orbital period:

$$T = \frac{2\pi r}{V_c} = \sqrt{\frac{3\pi}{G\rho}}$$

But for a test mass at radius r:

$$\frac{d^2 r}{dt^2} = -\frac{GM(r)}{r^2} = -\frac{4\pi G\rho}{3}\, r$$

(Harmonic oscillator with $\omega = 2\pi/T$)

Let's define the dynamical timescale as $T/4$. In this case:

$$t_{dyn} = T/4 = \sqrt{\frac{3\pi}{16 G\rho}} \sim \sqrt{1/G\rho}$$

<u>Potential:</u>

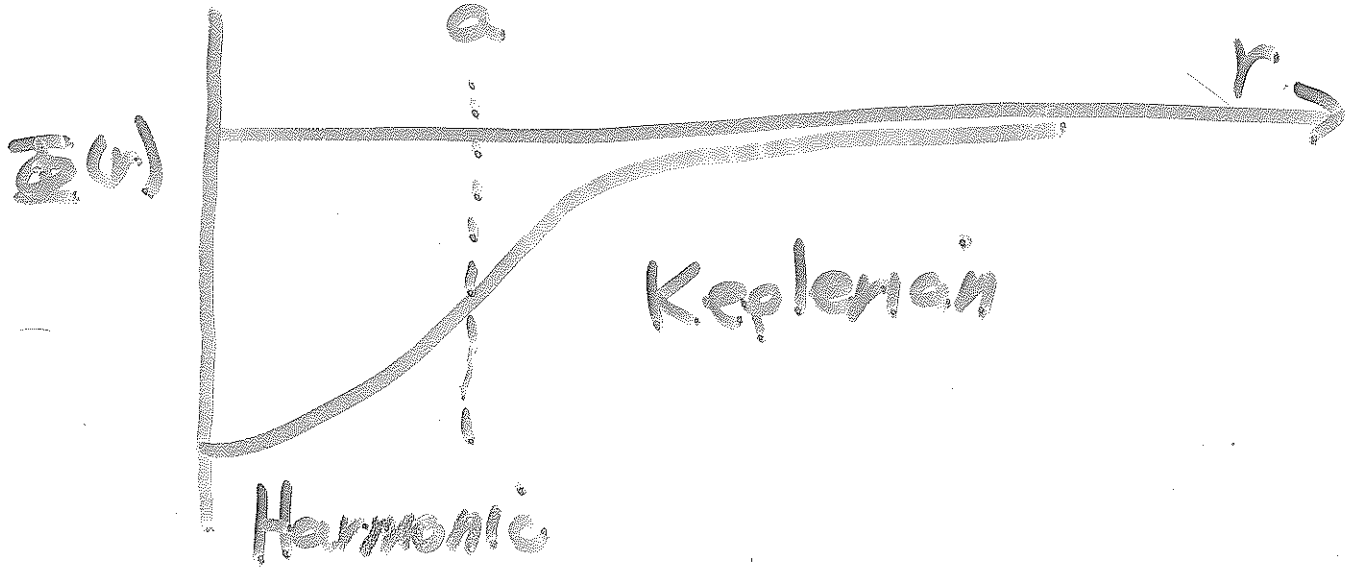
$$\Phi(r) = \begin{cases} -2\pi G\rho\left(a^2 - \tfrac{1}{3}r^2\right) & r < a \\ -\dfrac{4\pi G\rho a^3}{3r} & r > a \end{cases}$$

(recall potential = PE/(mass))

and since:

$$\Phi(r) = -4\pi G\left[\frac{1}{r}\int_0^r \rho(r')\,r'^2\,dr' + \int_r^\infty \rho(r')\,r'\,dr'\right]$$

$(\rho(r) = 0 \text{ for } r > a)$



<u>or:</u> $\Phi(r) = \Phi_0 + \frac{1}{2}\omega^2 r^2 \quad (r \le a)$

$= -\frac{GM}{r} \quad (r > a)$

Isochrone Potential

In order to avoid divergence a small r consider:

$$\Phi(r) = -\frac{GM}{b+\sqrt{b^2+r^2}}$$

Finite mass (M)

small r: $\Phi(r) \sim r^{-2}$ (const. density)

larger: $\Phi(r) \sim r^{-1}$

The corresponding circular velocity:

$$V_c^2(r) = \frac{GMr^2}{(b+a)^2 a} \quad \text{where } a \equiv \sqrt{b^2+r^2}$$

At larger $V_c = \sqrt{\frac{GM}{r}}$ (finite mass)

and

$$\rho(r) = \frac{1}{4\pi G}\frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi}{dr}\right)$$

$$= M\left[\frac{3(b+a)a^2 - r^2(b+3a)}{4\pi(b+a)^3 a^3}\right]$$

$$\rho(0) = \frac{M}{16\pi b^3}$$

Power-law Density Profile

$$\rho(r) = \rho_0 \left(r_0/r\right)^{\alpha}$$

<u>Note</u> if $\alpha < 3 \Rightarrow m(r)$ is ∞ for some (large) r

$$m(r) = \frac{4\pi \rho_0 r_0^{\alpha}}{3-\alpha} r^{3-\alpha}$$

The corresponding circular velocity:

$$V_c^2(r) = \frac{G 4\pi \rho_0 r_0^{\alpha}}{3-\alpha} r^{2-\alpha}$$

2 Interesting cases:

$\alpha = 0 \rightarrow$ harmonic potential

$\alpha = 2 \rightarrow$ singular isothermal sphere (V = constant)

$$V_e^2(r) = 2\int_r^{\infty} \frac{G m(r')}{r'^2} dr' = \frac{8\pi G \rho_0 r_0^{\alpha}}{(3-\alpha)(\alpha-2)} r^{(2-\alpha)}$$

$$= 2\frac{V_c^2(r)}{\alpha - 2} \quad (\alpha > 2)$$

Modified Hubble Profile

a useful modification of the standard Hubble profile is:

$$j_H(r) = \frac{j_0}{\left[1+(r/a)^2\right]^{3/2}}$$ (j_H is lumin. density)

with constant $m/L = \Upsilon$

$$M_H(r) = 4\pi a^3 \Upsilon j_0 \left\{ \ln\left[r/a + \sqrt{\frac{r^2}{a^2}+1}\right] - \frac{r}{a}\left(\frac{r^2}{a^2}+1\right)^{-1/2} \right\}$$

with:

$$\Phi_H(r) = -\frac{G M_H(r)}{r} - \frac{4\pi G \Upsilon j_0 a^2}{\sqrt{1+(r/a)^2}}$$

Note that m diverges logarithmically

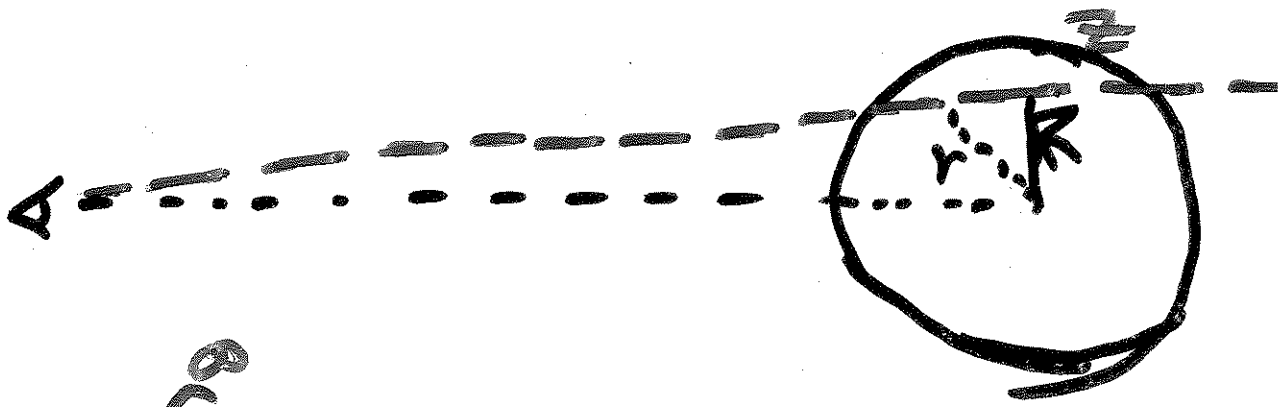
$$M_H(r) \simeq 4\pi a^3 \Upsilon j_0 \left[\ln(2r/a) - 1\right] \qquad r \gg a$$

But Φ_H is finite

Note on Projected Surface Density vs. Volume Density

In general we observe a projection of the density onto the plane of the sky.

Specifically:



$$I(R) = 2\int_0^\infty j(r)\,dz \qquad z = \sqrt{r^2 - R^2}$$

Abel's Integral equation (BM A1.4)

$$f(x) = \int_0^x \frac{g(t)\,dt}{(x-t)^\alpha}$$

So given $j(r)$ we can compute $I(R)$ but we observe $I(R)$. To get $j(r)$ from $I(R)$ is difficult and non-unique.

See Simonneau & Prada (2004, Rev. of Mex. Aston & Astroph. 40, 69) for solutions to Abel's integral for Sersic profiles ($e^{-R/n}$)

Now consider a Plummer Model

Spherical potential:

$$\Phi_p = \frac{-GM}{\sqrt{r^2+b^2}}$$

so:

$$\nabla^2 \Phi_p = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d\Phi_p}{dr}\right) = \frac{3GMb^2}{(r^2+b^2)^{5/2}}$$

and so:

$$\rho_p(r) = \left(\frac{3M}{4\pi b^3}\right)\left(1+\frac{r^2}{b^2}\right)^{-5/2}$$

A modification of this potential is called a Kuzmin potential:

$$\Phi_K(R,z) = \frac{-GM}{\sqrt{R^2+(a+|z|)^2}}$$

From symmetry arguments this reduces to a flat plane so

$\nabla^2 \Phi_K = 0$ everywhere except on plane $(z=0)$

Applying Gauss' theorm yields:

$$\Sigma_K(R) = \frac{aM}{2\pi(R^2+a^2)^{3/2}}$$

However a slight modification provides a good 1st approximation to a disk galaxy. Miyamoto-Nagai:

$$\Phi_M(R,z) = \frac{-GM}{\sqrt{R^2+(a+\sqrt{z^2+b^2})^2}}$$

$a=0 \rightarrow \Phi_m =$ Plummer's

$b=0 \rightarrow \Phi_m =$ Kuzmin's

Varying a and b produces a spherical "bulge" + flattened "disk".

$$\rho_m(R,z) = \left(\frac{b^2M}{4\pi}\right)\frac{aR^2+(a+3\sqrt{z^2+b^2})(a+\sqrt{z^2+b^2})^2}{\left[R^2+(a+\sqrt{z^2+b^2})^2\right]^{5/2}(z^2+b^2)^{3/2}}$$

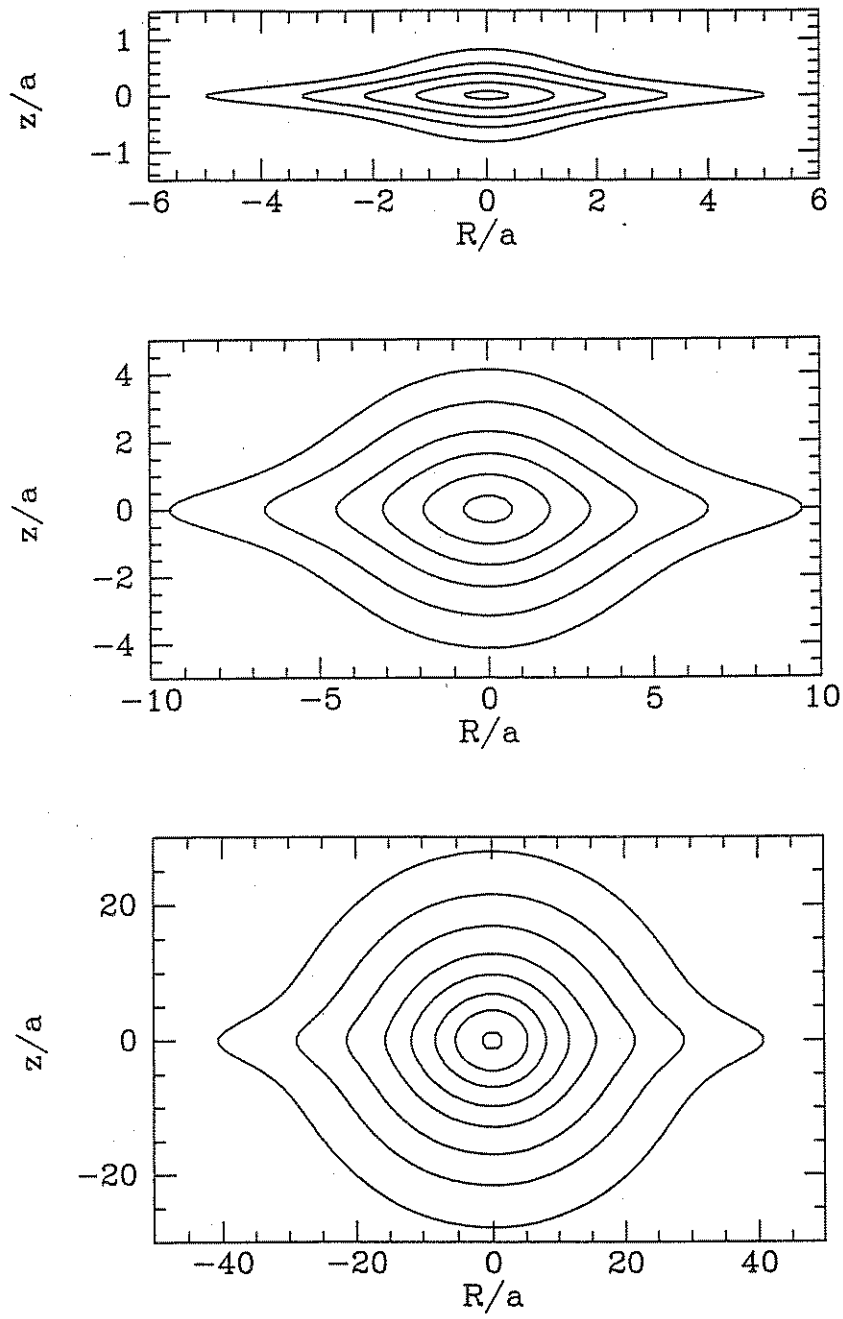


Figure 2-6. Contours of equal density in the (R, z) plane for the Miyamoto-Nagai density distribution (2-50b) when: $b/a = 0.2$ (top); $b/a = 1$ (middle); $b/a = 5$ (bottom). Contour levels are: $(0.3, 0.1, 0.03, 0.01, \ldots)M/a^3$ (top); $(0.03, \ldots)M/a^3$ (middle); $(0.001, \ldots)M/a^3$ (bottom).

Consider Poisson's Equation for Very Flattened Axisymmetric Systems

In cylindrical coords. Poisson's Equation:

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right)+\cancel{\frac{1}{R^2}\frac{\partial^2 \Phi}{\partial \phi^2}}^{\,0}+\frac{\partial^2 \Phi}{\partial z^2}=4\pi G\rho(R,z)$$

so, since $F_R = -\frac{\partial \Phi}{\partial R}$ we have:

$$\frac{\partial^2 \Phi}{\partial z^2}=4\pi G\rho(R,z)+\frac{1}{R}\frac{\partial}{\partial R}(RF_R)$$

If we examine the Miyamoto-Nagai potential we see that as $b \to 0$ (more and more flattened) the density ($@\, z=0$) rises as b^{-1}. However, at $b=0$ the radial force $(F_R) = -\partial \Phi_K/\partial R$, where Φ_K is the Kuzmin potential. So, near $z \simeq 0$ and we have:

$$\frac{\partial^2 \Phi(R,\cancel{z})}{\partial z^2}\overset{\sim 0}{\simeq} 4\pi G\rho(R,\cancel{z})^{\sim 0}$$

The vertical variation in the potential at a given radius depends only on the density distribution at that radius.

So Poisson's equation can be solved in two steps: 1) assume a disk of zero thickness and solve for $\Phi(R,0)$, 2) at each R solve above equation for $\Phi(R,z)$.

So for a thin disk we can separate the radial and vertical potentials.

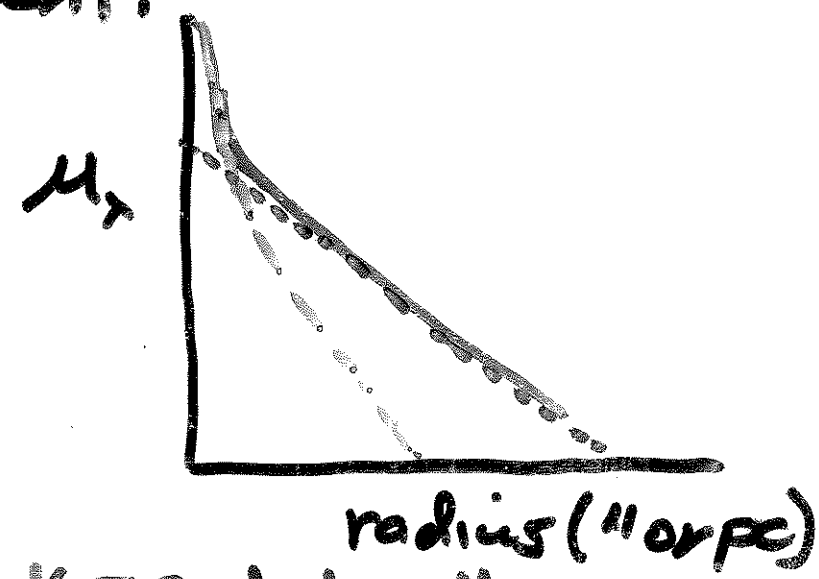
Note that in general an solution must be found numerically

Section 2.4 in BT discusses a Multipole expansion for the general problem. The solution is obtained from Laplace's equation applied to a series of thin shells.

The result involves a series of associated Legendre polynomials or spherical harmonics.

Mass Modeling of Disk Galaxies

Recall that the surface photometry of Spiral galaxies reveals a bulge + disk component:



If we just look at the disk:

$$\mu(r) = \mu_0 + C r$$

or

$$\Sigma = \Sigma_0 e^{-\alpha r}$$

$1/\alpha$ = scale length

where $\alpha = 1.086\, C$

Early work (e.g. Freeman 1970, ApJ 160, 811) indicated a small range in μ_0 ($\langle \mu_0 \rangle_B = 21.65 \pm 0.3$ mag/arcsec^2)

Selection effects important:

Disney 1976 Nature 263, 573

Surface Photometry of Edge-On Systems Reveals Vertical Struct.

Van der Kruit & Searle (1981: AA 95, 105
AA 95, 116)

$$I(r,z) = I_0 e^{-r/r_0} \operatorname{sech}^2(z/z_0)$$

with r_0 and z_0 the radial and vertical scale heights, respectively.

Projections:

Face-On: $I(r) = 2 I_0 z_0 e^{-r/r_0}$

Edge-On: $I(r,z) = 2 I_0 r K_1(r/r_0) \operatorname{sech}^2(z/$

where K_1 is the modified Bessel function of the first kind

Note that the $\sec^2 h(z/z_0)$ dependence is expected for an isothermal ∞ sheet of stars.

Potential/Density of Disks

Toomre (1962, ApJ 138, 385) showed that a general approach for obtaining potential/density pairs for disks can be developed using Hankle transforms.

Laplace's equation in cylindrical coords

$$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \Phi}{\partial R}\right) + \frac{\partial^2 \Phi}{\partial z^2} = 0$$

If the disk is separated into radial and vertical components:

$$\Phi(R,z) = J(R)\,Z(z)$$

then separation of variables yields:

$$\frac{1}{J(R)R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right) = \frac{-1}{Z(z)}\frac{d^2 Z}{dz^2} = -k^2$$

real or complex number

or:

$$\frac{d^2 Z}{dz^2} - k^2 Z = 0$$

vertical

and:

$$\frac{1}{R}\frac{d}{dR}\left(R\frac{dJ}{dR}\right)+k^2 J(R)=0 \qquad \text{radial}$$

Integrating the vertical equation:

$$Z(z)=S\,e^{\pm kz} \qquad S \text{ is a const.}$$

Substituting $u=kR$ into radial eqn:

$$\frac{1}{u}\frac{d}{du}\left(u\frac{dJ}{du}\right)+J(u)=0$$

One Solution is a Bessel function (order 0).

SO: $\Phi(R,z)=J_0(kR)\,e^{\pm kz}$

Application of Gauss' Theorm and Poisson's equations gives:

$$\Sigma_k(R)=\frac{k}{2\pi G}J_0(kR)$$

But what about for an arbitrary $\Sigma(R)$? $\Phi(R,z)$, $\Sigma_k(R)$ potential/density pairs are related by Bessel functions.

For an arbitrary $\Sigma(R)$ but disk geometry (i.e. cylindrical symmetry) the relationship involves an <u>Integral Transform</u>. But Fourier transforms don't make sense — the disk Φ won't have wave-like properties. So instead of a kernal like $e^{i\theta}$ we will want to use a Bessel function. This is called a <u>Hankel transform</u>. Specifically:

$$\Sigma(R) = \frac{-1}{2\pi G}\int_0^\infty S(k)\,J_0(kR)\,k\,dk$$

and:

$$\Phi(R,z) = \int_0^\infty S(k)\,J_0(kR)\,e^{-k|z|}\,dk$$

Thus $S(k)$ is the Hankel Transform of $-2\pi G\,\Sigma(R)$. Applying the inversion relation for Hankel Transforms gives:

$$S(k) = -2\pi G \int_0^\infty J_0(kR)\,\Sigma(R)\,R\,dR$$

analogous to a Fourier Transform.

Eliminating $S(k)$ we get:

$$\Phi(R,z) = -2\pi G \int_0^\infty J_0(kR)\,e^{-k|z|}\,dk \int_0^\infty \Sigma(R')\,J_0(kR')\,R'\,dR'$$

For the circular velocity recall:

$$V_c^2(R) = R\left(\frac{\partial \Phi}{\partial R}\right)_{z=0}$$ so:

$$V_c^2(R) = -R\int_0^\infty S(k)\,J_1(kR)\,k\,dk$$

since

$$\frac{d\,J_0(x)}{dx} = -J_1(x)$$

Thus given an arbitrary $\Sigma(R)$ we can compute $S(k)$ and then $V_c(R)$!

Consider 2 Examples

1) Mestel's disk: $\Sigma(R) = \frac{\Sigma_0 R_0}{R}$

(density inversely proportional to radius)

$$S(k) = -2\pi G \Sigma_0 R_0 \int_0^\infty J_1(kR)\, dR$$

$$= \frac{-2\pi G \Sigma_0 R_0}{k}$$

Thus:

$$V_c^2(R) = 2\pi G \Sigma_0 R_0 R \int_0^\infty J_1(kR)\, dk$$

$$= 2\pi G \Sigma_0 R_0$$

Rotational velocity of Mestel's disk is constant.

and since

$$M(R) = 2\pi \int_0^R \Sigma(R') R'\, dR'$$

we see that:

$$V_c^2(R) = \frac{G M(R)}{R}$$

2) An exponential disk $\Sigma(R) = \Sigma_0 e^{-R/R_0}$

R_0 – scale length (Y/a)

It can be shown that in this case:

$$S(k) = \frac{-2\pi G \Sigma_0 R_0^2}{\left[1 + (kR_0)^2\right]^{3/2}}$$

So for the potential we have:

$$\Phi(R,z) = -2\pi G \Sigma_0 R_0^2 \int_0^\infty \frac{J_0(kR)\, e^{-k|z|}}{\left[1 + (kR_0)^2\right]^{3/2}}\, dk$$

for the case where $z=0$ this yields:

$$\Phi(R,0) = -\pi G \Sigma_0 R \left[I_0(y) K_1(y) - I_1(y) K_0(y)\right]$$

where $y \equiv R/2R_0$ and I_n and K_n are the the modified Bessel functions of the 1st and 2nd kinds.

If we differentiate with respect to R:

$$V_c^2(R) = R\frac{\partial \Phi}{\partial R} = 4\pi G \Sigma_0 R_0 y^2 \left[I_0(y) K_0(y) - I_1(y) K_1(y)\right]$$

and

$$M(<R) = 2\pi \Sigma_0 R_0^2 \left[1 - \exp(-R/R_0)(1 + R/R_0)\right]$$

Note that in principal one could compute $\Sigma(R)$ given $V_c^2(R)$. However one needs to know $\frac{dV_c^2}{dR}$ very accurately and the inversion is very unstable. It is better to assume a form for $\Sigma(R)$ and then match the predicted $V_c^2(R)$ to the observations.

A Specific Example:

Mass Modeling of NGC 3198 (van Albada et al. 1985, ApJ 295, 305)

Data:

- Surface photometry
- High Resolution HI velocity Field

Model:

- Rotation curve from Expon. Disk

$$\frac{V^2(R)}{R} = \pi G \Upsilon \mu_0 \alpha R (I_0 K_0 - I_1 K_1)$$

Find maximum Υ that fits inner rot. curve (maximum disk)

The residuals are assumed to arise from a massive dark halo.

<u>Assume</u>:

$$\rho_H(R) = \rho_H(0)\left[1+(R/R_c)^{\gamma}\right]^{-1}$$

<u>so (spherical symmetry)</u>:

$$V_H^2(R) = \frac{G\,m(<R)}{R} \qquad m(<R) = \int_0^R 4\pi r^2 dr\, \rho_H$$

<u>Note</u>: all fits assume $\Upsilon \neq f(R)$

- disk dominates at small R
- halo dominates at large R

<u>Homework</u>: write a program to model the N3198 data. Show V_R contribution from disk and halo

$\Upsilon = ?$ What is $\frac{m_H}{m_D}$ vs R?

What is R_c? (adopt params. from optical photom.)

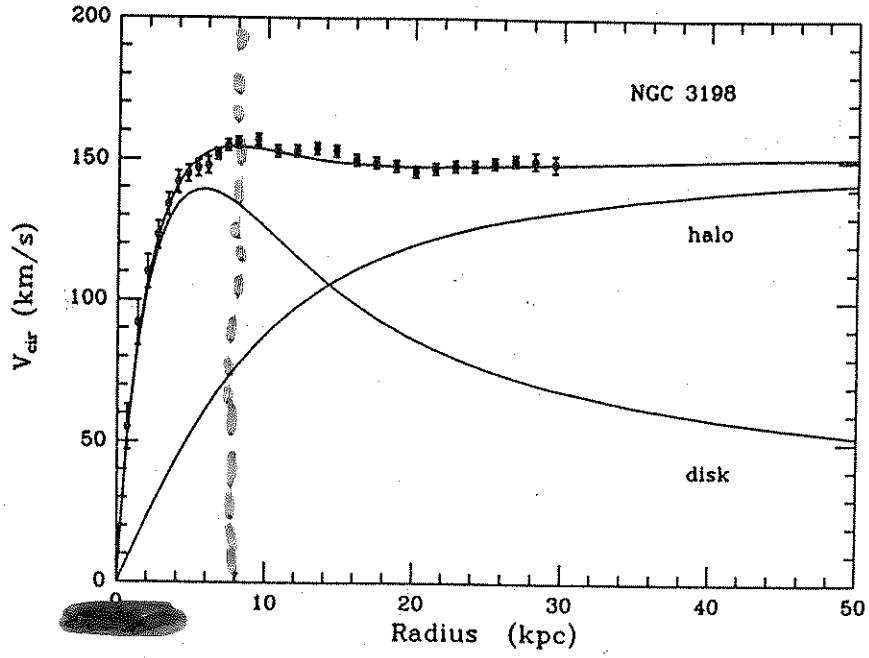


FIG. 4.—Fit of exponential disk with maximum mass and halo to observed rotation curve (*dots with error bars*). The scale length of the disk has been taken equal to that of the light distribution (60″, corresponding to 2.68 kpc). The halo curve is based on eq. (1), $a = 8.5$ kpc, $\gamma = 2.1$, $\rho(R_0) = 0.0040\ M_\odot\ \mathrm{pc}^{-3}$.

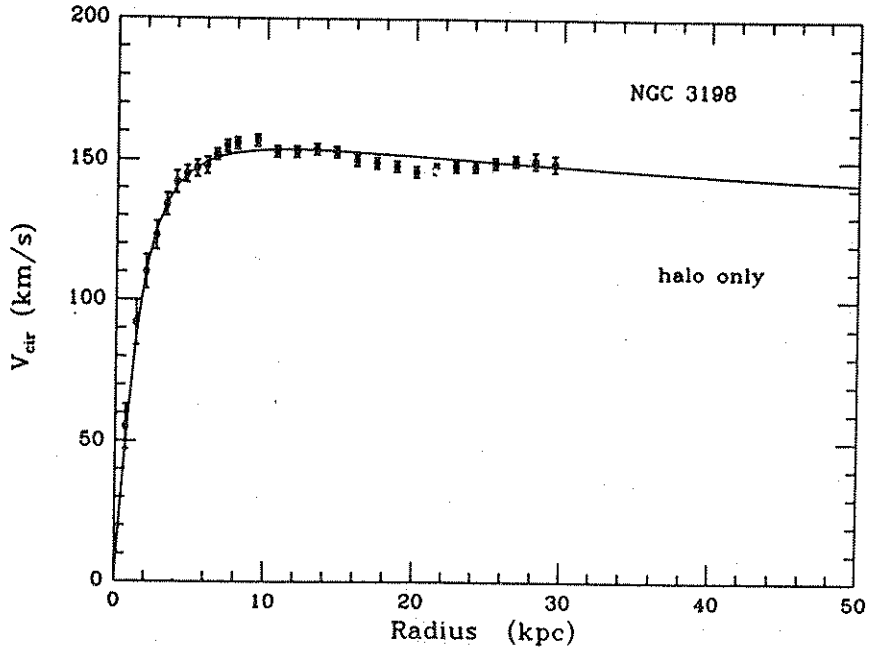


FIG. 8.—Fit of halo without disk; $a = 1.5$ kpc, $\gamma = 2.25$, $\rho(R_0) = 0.0074\ M_\odot\ \mathrm{pc}^{-3}$.

Orbits of Stars (Ch. 3 BT)

What kinds of orbits are possible in a given potential?

How do stars move in complex potentials?

Recall that in most systems the # of stars is large ($N > 10^{11}$) so we can consider/assume the potential is smooth.

Spherical Potentials

Let the position vector of a star be $\vec{r} = r\vec{e}_r$

The radial force/mass is:

$$\vec{F} = F(r)\vec{e}_r$$

and the equation of motion is:

$$\frac{d^2\vec{r}}{dt^2} = F(r)\vec{e}_r$$

Now consider the angular momentum

$$\vec{L} = \vec{r} \times \frac{d\vec{r}}{dt}$$

so we take the cross product of $\vec{r}$ and each side of the above equation:

$$\vec{r} \times \frac{d^2\vec{r}}{dt^2} = \vec{r} \times F(r)\vec{e}_r$$

or:

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right) = F(r)\,\vec{r} \times \vec{e}_r$$

but $\vec{r} \times \vec{e}_r = 0$ so

$$\frac{d}{dt}(\vec{L}) = 0$$ and so we conclude

$\vec{L} = \text{constant}$ (ang. mom. is conserved)

$\therefore$ star must move in a plane and so we can use polar coord. (r, ψ).

Now recall that:

$$\frac{d\vec{e}_r}{d\psi} = \vec{e}_\psi \quad \text{and} \quad \frac{d\vec{e}_\psi}{d\psi} = -\vec{e}_r$$

so since

$$\frac{d\vec{e}_r}{dt} = \dot{\psi}\,\vec{e}_\psi \quad \text{and} \quad \frac{d\vec{e}_\psi}{dt} = -\dot{\psi}\,\vec{e}_r$$

we then write the equation of motion:

$$\ddot{r} - r\dot{\psi}^2 = F(r)$$

$$2\dot{r}\dot{\psi} + r\ddot{\psi} = 0 \qquad \text{ang. mom.}$$

multiplying the second equation by r and integrating gives:

$$r^2\dot{\psi} = \text{constant} = L \qquad \left(\begin{array}{c}\text{equal areas}\\ \text{in equal time}\end{array}\right)$$

This can be rewritten as:

$$\frac{d\psi}{dt} = \frac{L}{r^2}$$

or:

$$\frac{d}{dt} = \frac{L}{r^2}\frac{d}{d\psi}$$

so we can eliminate t, replacing with ψ; and the equation of motion becomes:

$$\frac{L^2}{r^2}\frac{d}{d\psi}\left(\frac{1}{r^2}\frac{dr}{d\psi}\right) - \frac{L^2}{r^3} = F(r)$$

If we substitute $u \equiv 1/r$ we have:

$$\frac{d^2u}{d\psi^2} + u = \frac{-F(1/u)}{L^2u^2}$$

In general this equation must be solved numerically. However we can gain insight by multiplying by $\frac{du}{d\psi}$ and integrating wrt ψ to obtain:

$$\left(\frac{du}{d\psi}\right)^2 + \frac{2\Phi}{L^2} + u^2 = \text{const.} \equiv \frac{2E}{L^2}$$

since $F(r) = -\frac{d\Phi}{dr} = u^2 \frac{d\Phi}{du}$

Thus:

$$E = \tfrac{1}{2}\left(\frac{dr}{dt}\right)^2 + \tfrac{1}{2}\left(r\frac{d\psi}{dt}\right)^2 + \Phi(r)$$

and E is the total energy per unit mass

Note that for bound orbits $\frac{du}{d\psi} = 0$

so we have:

$$u^2 + \frac{2[\Phi(1/u) - E]}{L^2} = 0$$

So the roots will be u_1 and u_2

these are the pericenter(r_1) and the apocenter(r_2)

We use $L = r^2\dot{\psi}$ to eliminate $\dot{\psi}$ so:

$$\left(\frac{dr}{dt}\right)^2 = 2(E-\Phi) - \frac{L^2}{r^2} \quad \text{or:}$$

$$\frac{dr}{dt} = \pm\sqrt{2(E-\Phi) - L^2/r^2}$$

The radial period is thus:

$$T_r = 2\int_{r_1}^{r_2} \frac{dr}{\sqrt{2(E-\Phi)-L^2/r^2}}$$

during which:

$$\Delta\psi = 2\int_{r_1}^{r_2} \frac{d\psi}{dr}\,dr = 2\int_{r_1}^{r_2} \frac{L}{r^2}\frac{dt}{dr}\,dr$$

Substituting from above yields:

$$\Delta\psi = 2L\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2(E-\Phi)-L^2/r^2}}$$

Note that

$$T_\psi = \frac{2\pi}{\Delta\psi}T_r \quad \text{but in general}$$

$\Delta\psi/2\pi$ will not be rational.

and so the orbit is not closed and the star traces a rosette pattern. Note that as the pattern is filled in r_1 and r_2 (the pericenter and apocenter) become obvious.

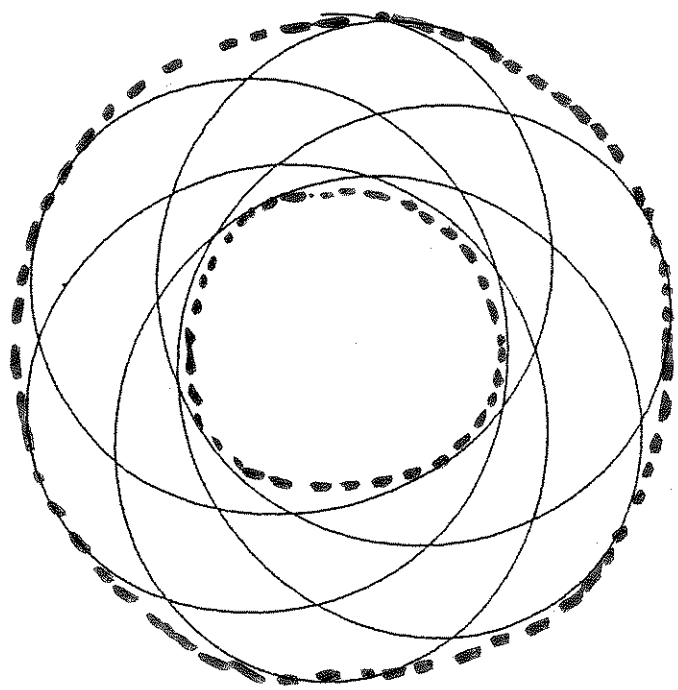


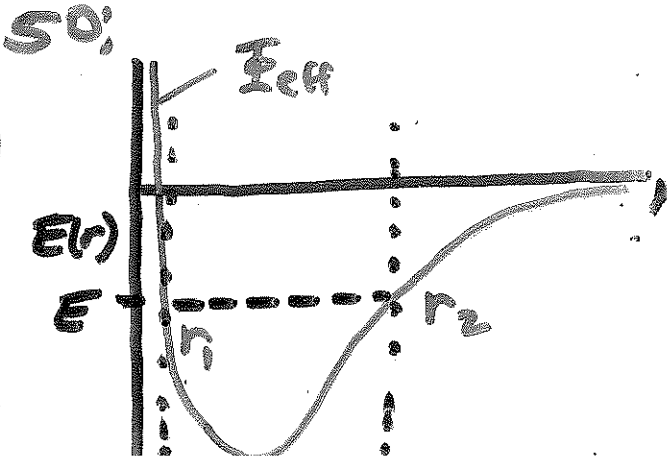
Figure 3-1. A typical orbit in a spherical potential forms a rosette.

We can define an effective potential:

$$\Phi_{eff}(r) = \Phi(r) + L^2/2r^2$$

so:

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 = E - \Phi_{eff}(r)$$



Stellar Orbits in Spherical Potentials

a.) Spherical Harmonic Potential

Recall the form for a Harmonic Potential:

let

$$\Phi(r) = \tfrac{1}{2}\Omega^2 r^2 + \text{constant}$$

the corresponding density distribution:

$$\rho(r) = \text{constant} \qquad r \leq r_0$$
$$\rho(r) = 0 \qquad r > r_0$$

In this case the equation of motion has an analytic solution:

<u>Cartesian coords:</u>

$$\ddot{x} = -\Omega^2 x \qquad \ddot{y} = -\Omega^2 y$$

Note that this is the equation for a <u>Harmonic</u> <u>Oscillator</u>. The solution is:

$$x = X\cos(\Omega t + \varepsilon_x) \quad y = Y\cos(\Omega t + \varepsilon_y)$$

where X, Y, and ε_x, ε_y are arbitrary constants. Each orbit is an ellipse and closed since $T_x = T_y$.

b) Keplerian Potential

Recall that outside any spherical mass distribution the potential is equivalent to that of a point mass. That is, under the influence of an inverse-square law force:

$$F(r) = \frac{-GM}{r^2} \quad \text{or} \quad F(u) = -GMu^2$$

so the equation of motion becomes:

$$\frac{d^2u}{d\psi^2} + u = \frac{GM}{L^2}$$

The general solution is:

$$u(\psi) = C\cos(\psi - \psi_0) + \frac{GM}{L^2}$$

where $C > 0$ and ψ are arbitrary const.

If we define the orbit's eccentricity:

$$e \equiv \frac{CL^2}{GM}$$

and the semi-major axis as:

$$a \equiv \frac{L^2}{GM(1-e^2)}$$

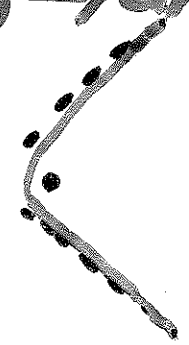
then the equation of motion becomes:

$$r(\psi) = \frac{a(1-e^2)}{1+e\cos(\psi-\psi_0)}$$

$e>1$: hyperbola
$e=1$: parabola
$0<e<1$: ellipse
$e=0$: circle

Note that for $e>1$ orbits are unbound since $r \to \infty$ as $(\psi-\psi_0) \to \arccos(-1/e)$.
Also as $r \to \infty$ we have

$$e^2 = 1 + \left(\frac{L V_\infty}{GM}\right)^2$$



For $e<1$ all orbits are bound (r is finite for all ψ) and periodic (closed). Differentiation of the equation of motion in terms of time (i.e. rewriting as $r(t)$) and setting $\dot{r}=0$ yields the **pericenter** and **apocenter**:

$$r_1 = a(1-e) \quad \text{and} \quad r_2 = a(1+e)$$

The azimuthal and radial periods are the same since the orbit is closed:

$$T_r = T_\psi = 2\pi\sqrt{\frac{a^3}{GM}}$$

Note: For an extended mass distribution $\frac{T_r}{T_\psi} < 1$ so orbits precess. This can be shown using an Isochrone potential (see BT sec. 3.1c for an analytic solution).

Phase Space and Integrals of Motion

Recall that for an arbitrary potential the equation of motion must be numerically integrated. However, some insight into the nature/behavior of orbits can be obtained by examining their location in phase space.

Let the distribution function of stars be defined as the probability of a star having a given spacial

position (x_1, x_2, x_3) and velocity (v_1, v_2, v_3). This 6-d space specifies the motion of a given star since we can integrate the equations of motion forward or backward in time. Although the motion (degree of freedom) of any star is somewhat unconstrained there are conserved quantities that limit the degrees of freedom for a particular star.

<u>Integral of Motion</u>: any function of phase-space coords. that is constant along an orbit:

$$I[\vec{x}(t_1), \vec{v}(t_1)] = I[\vec{x}(t_2), \vec{v}(t_2)]$$

Energy (E) is always an Integral of Motion. If the potential is spherically symmetric then so is angular momentum (L).

But even in the case of spherical symmetry we have only 4 I of M and we need 6 to fully specify the motion of the star. It can be shown that the initial orbital phase (ψ_0) is a 5th I of M (BT 3.1)

Axisymmetric Potentials

The situation for axisymmetric Φ is more complex. Here only $L_z = R^2\dot{\phi}$ is conserved so only 2 I of M exist.

Recall for cylindrical coords:

$$\frac{d^2\vec{r}}{dt^2} = -\nabla\Phi(R, z)$$

Thus:

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial\Phi}{\partial R}$$ radial equ. of motion

$$\frac{d}{dt}(R^2\dot{\phi}) = 0$$ conserv. of $\vec{L}_z$

$$\ddot{z} = -\frac{\partial\Phi}{\partial z}$$ z equ. of motion

Using $\vec{L}_z$ conservation to eliminate $\dot{\phi}$:

$$\ddot{R} = \frac{-\partial \Phi_{eff}}{\partial R}, \quad \ddot{z} = \frac{-\partial \Phi_{eff}}{\partial z}$$

where $$\Phi_{eff} \equiv \Phi(R,z) + \frac{L_z^2}{2R^2}$$

Consider the effective potential

$$\Phi_{eff} = \tfrac{1}{2} V_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right) + \frac{L_z^2}{2R^2}$$

(potential for star in oblate spheroid with constant V_0)

The contours in z vs. R show a minimum:

$$\frac{\partial \Phi_{eff}}{\partial R} = \frac{\partial \Phi}{\partial R} - \frac{L_z^2}{R^3} = 0$$

$$\frac{\partial \Phi_{eff}}{\partial z} = 0$$

So at $R_g = R_{min}$:

$$\frac{L_z^2}{R_g^3} = R_g \dot{\phi}^2$$ — a circular orbit with L_z

$\Phi_{eff}(R_g)$ is the energy of this circular orbit. In fact the energy in an arbitrary orbit is:

$$E = \tfrac{1}{2}\left[\dot{R}^2 + (R\dot{\phi})^2 + \dot{z}^2\right] = \tfrac{1}{2}(\dot{R}^2 + \dot{z}^2) + \left(\Phi + \frac{L_z^2}{2R^2}\right)$$

$$= \tfrac{1}{2}(\dot{R}^2 + \dot{z}^2) + \Phi_{eff}$$

So Φ_{eff} is the sum of Φ and kinetic energy in ϕ direction.

The difference $E - \Phi_{eff} = KE(R, z)$

Since $E \geq \Phi_{eff}$ the curve bounding this region is called the zero-velocity curve.

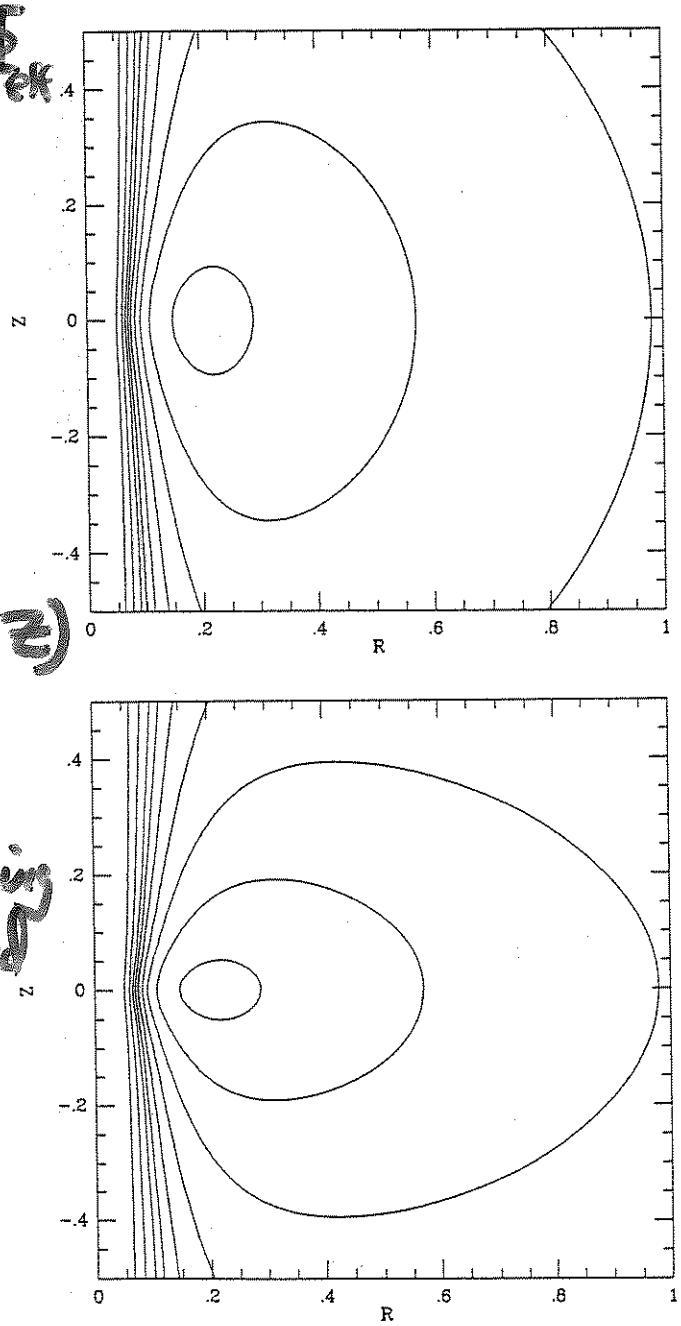


Figure 3-2. Level contours of the effective potential of equation (3-50) when $L_z = 0.2$ and: $q = 0.9$ (top); $q = 0.5$ (bottom). Contours are shown for $\Phi_{\text{eff}} = -1, -0.5, 0, 0.5, 1, 1.5, 2, 3, 5$, assuming $v_0 = 1$.

Given a Φ we can numerically integrate eqn. of motion for various initial conditions and follow $R(t)$, $z(t)$. Projections of the phase space $(R, z, \dot{R}, \dot{z})$ reveal "families of orbits."

Example potential given above

$$\Phi = \tfrac{1}{2} v_0^2 \ln\left(R^2 + \frac{z^2}{q^2}\right)$$

can be used to show orbits projected onto R-Z plane. In R-$\dot{R}$ plane the star is plotted when crossing Z plane ($z=0$, $\dot{z}>0$)

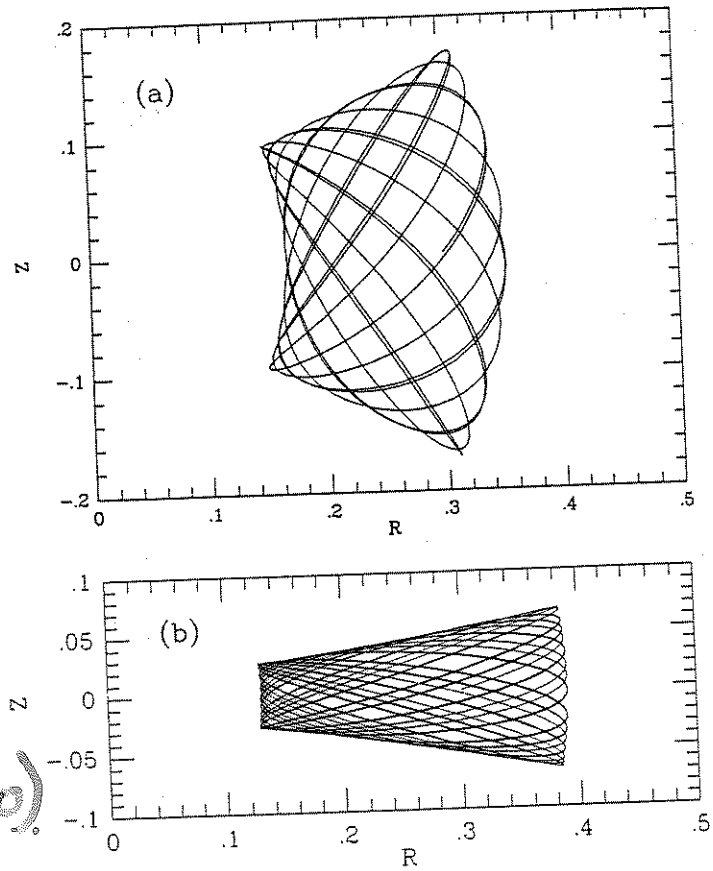


Figure 3-3. Two orbits in the potential of equation (3-50) with $q = 0.9$. Both orbits are at energy $E = -0.8$ and angular momentum $L_z = 0.2$, and we assume $v_0 = 1$.

Poincaré surface of section.

The area is bounded by the curve

$$E \geq \tfrac{1}{2}\dot{R}^2 + \Phi_{eff}.$$

Note that although L is not precisly conserved it almost is even though the potential is somewhat flattened.

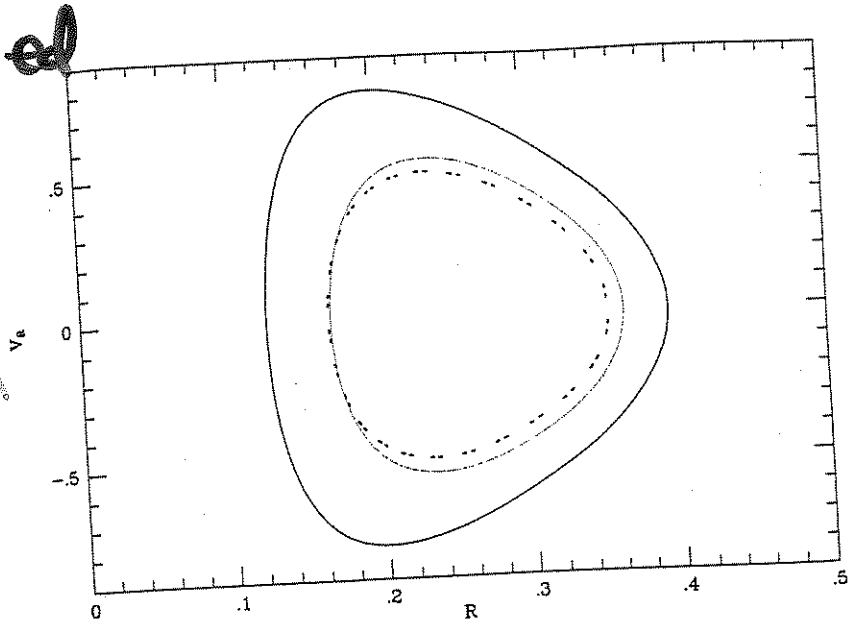


Figure 3-4. Points generated by the orbit of Figure 3-3a in the $(R, \dot{R})$ surface of section. If the total angular momentum L of the orbit were conserved, the points would fall on the dotted curve. The full curve is the zero-velocity curve at the energy of this orbit.

L approximately conserved.

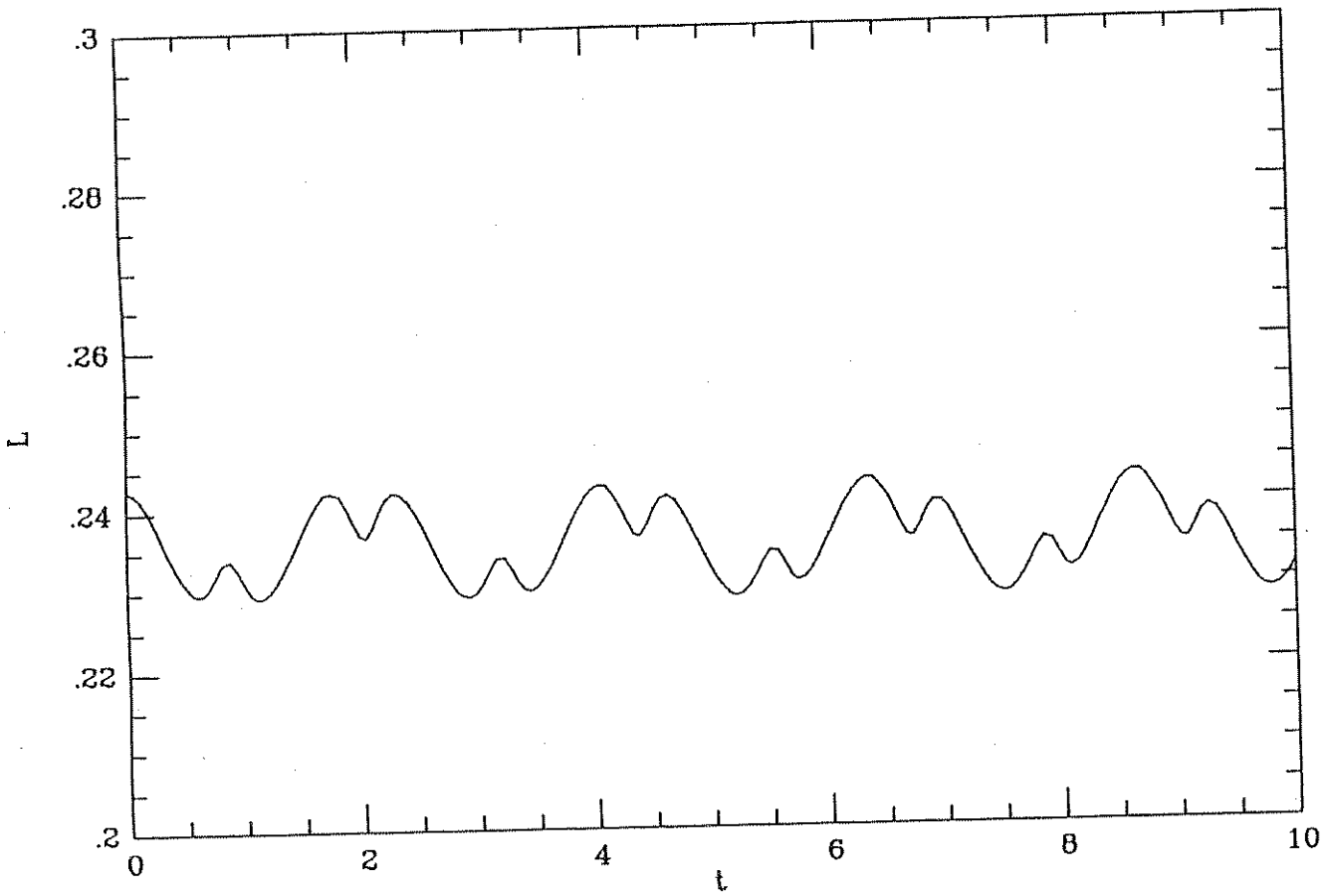


Figure 3-5. The total angular momentum is almost constant along the orbit shown in Figure 3-4.

Non axisymmetric Potentials

- useful for modeling barred galaxies.

$\Phi(x,y) = \frac{1}{2} v_0^2 \ln\left(R_c^2 + x^2 + y^2/q^2\right) \quad q \leq 1$

The simplest orbits can be grouped into two classes:

- loop orbits
- box orbits

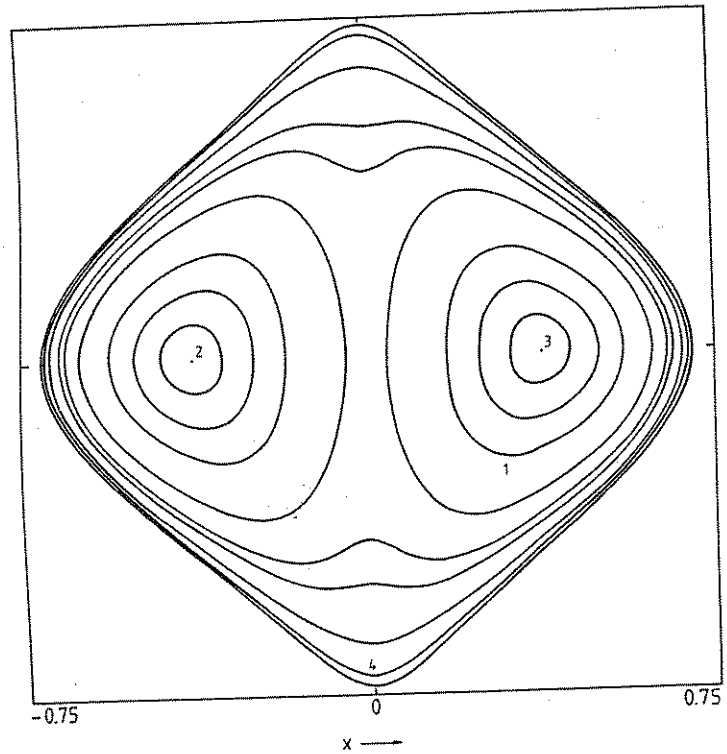


Figure 3-8. The $(x, \dot{x})$ surface of section formed by orbits in Φ_L of the same energy as the orbits depicted in Figure 3-7. The isopotential surface of this energy cuts the long axis at $x = 0.7$.

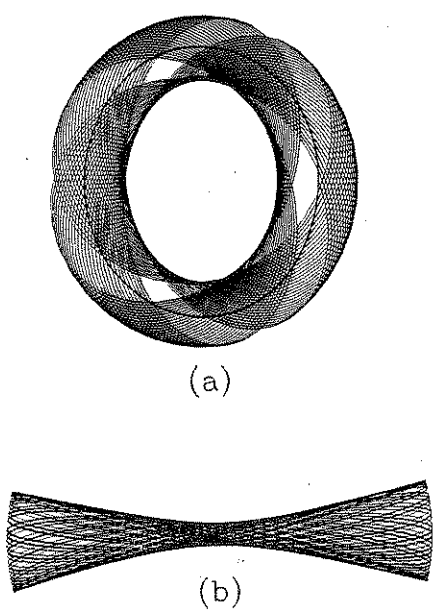


Figure 3-7. Two orbits of a common energy, $E = -0.337$, in the potential Φ_L of equation (3-77) when $v_0 = 1$, $q = 0.9$ and $R_c = 0.14$: (a) a loop orbit; (b) a box orbit. The closed parent of the loop orbits is superposed on the orbit of (a).