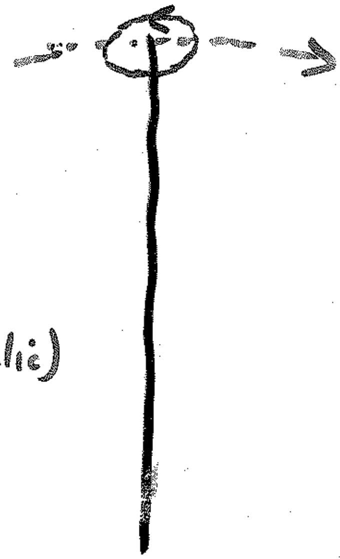
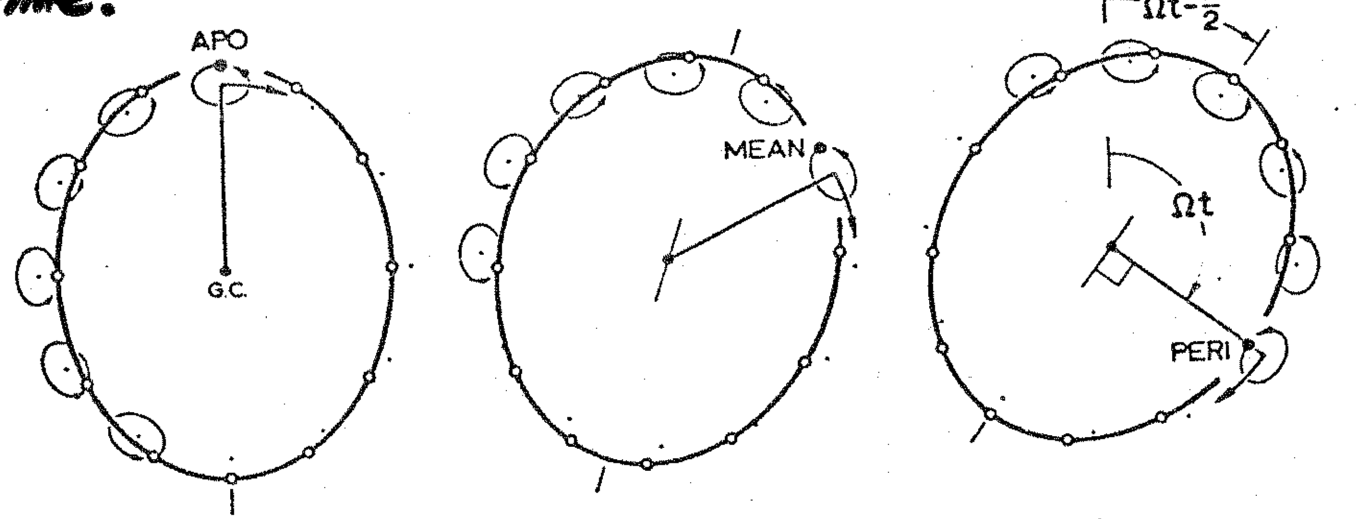
Importance of Epicycles

let K be epicyclic ~~period~~ frequency of oscilation about a mean ellipical orbit.



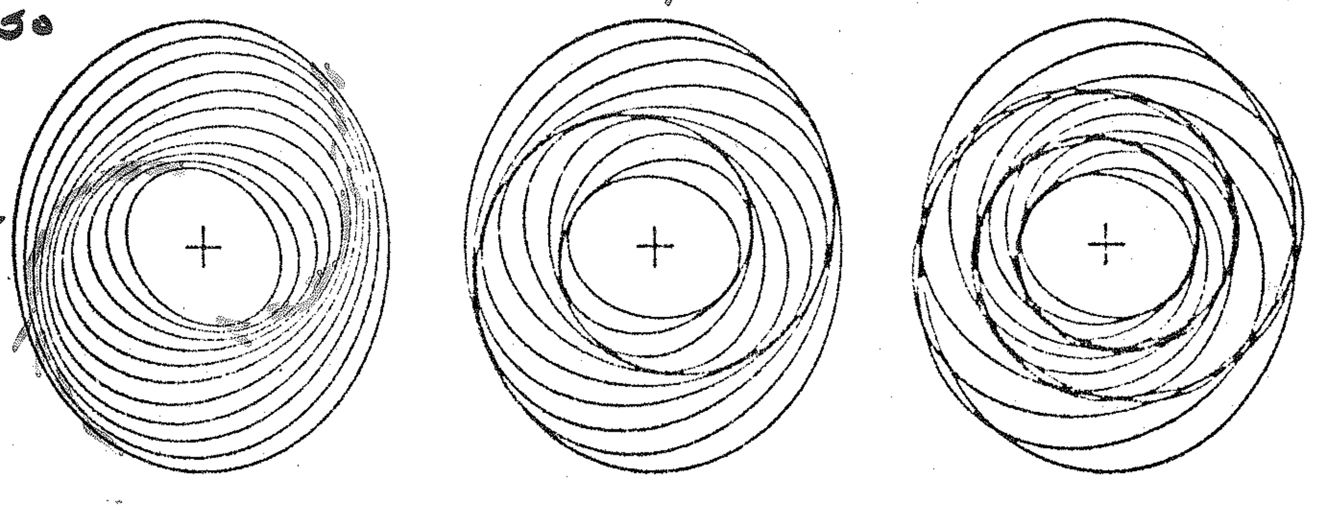
Between apogalactrion & perigalactron the particle moves through ½ orbit (epicyclic) $(\kappa t = \pi)$ But there is a phase difference for other particles around the mean ellipse so it precesses by $(\Omega - \kappa/2)t$ in same time.



$\kappa t = 0$ $\pi/2$ $\kappa t = \pi$

Introduce a phase difference vs radius.

$\Omega \propto r$ so different ellipses precess at different rates



a = 5 a = 10 a = 16.7

Trick is to "tune" precession rates to all have same period Ω_p

Since $\Omega_p = \Omega - \kappa/2$ is ~ constant over a large range in radius one only needs to introduce a phase difference vs. radius (i.e. impose a pattern) and it will subsequently be maintained.

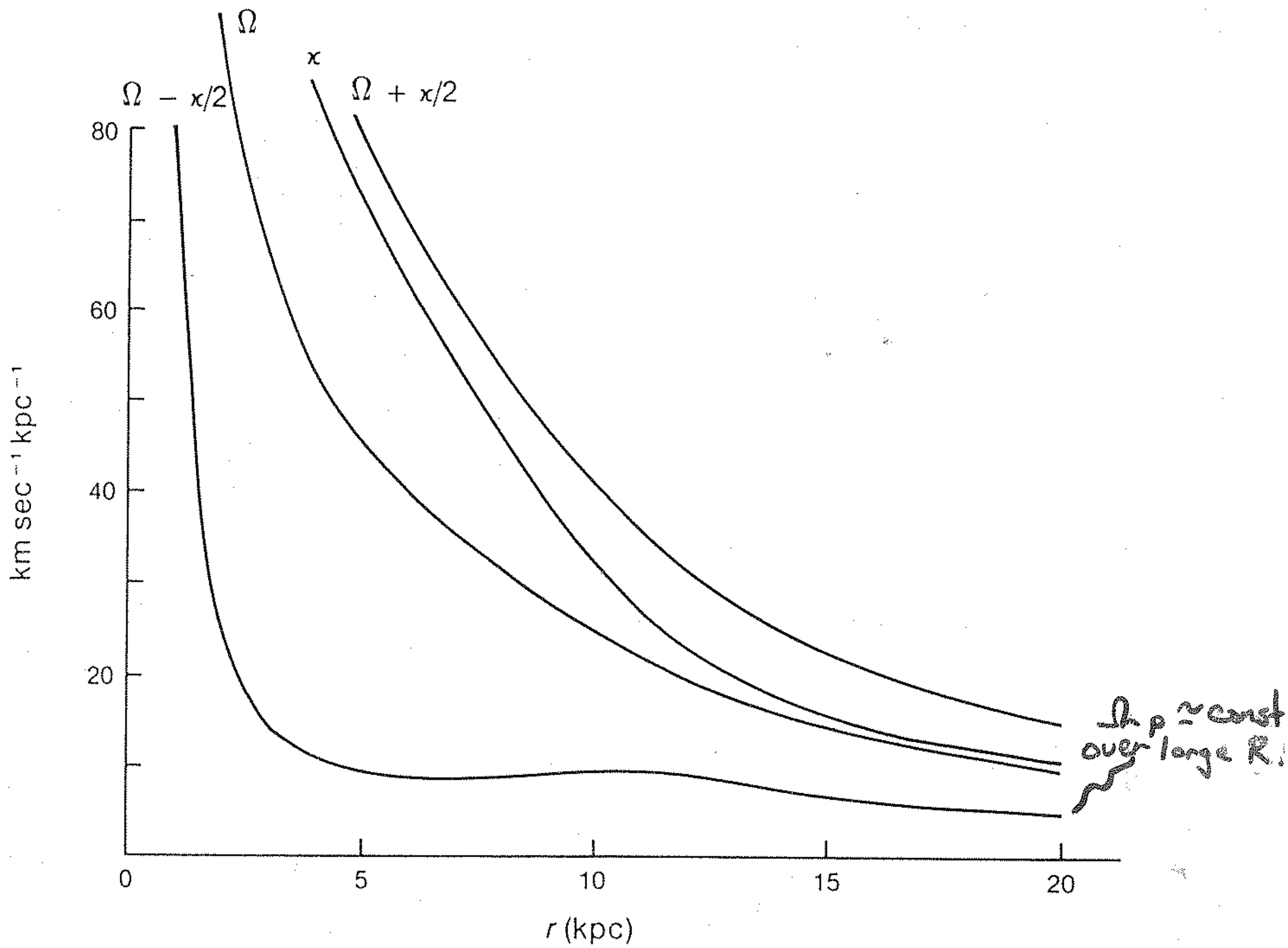
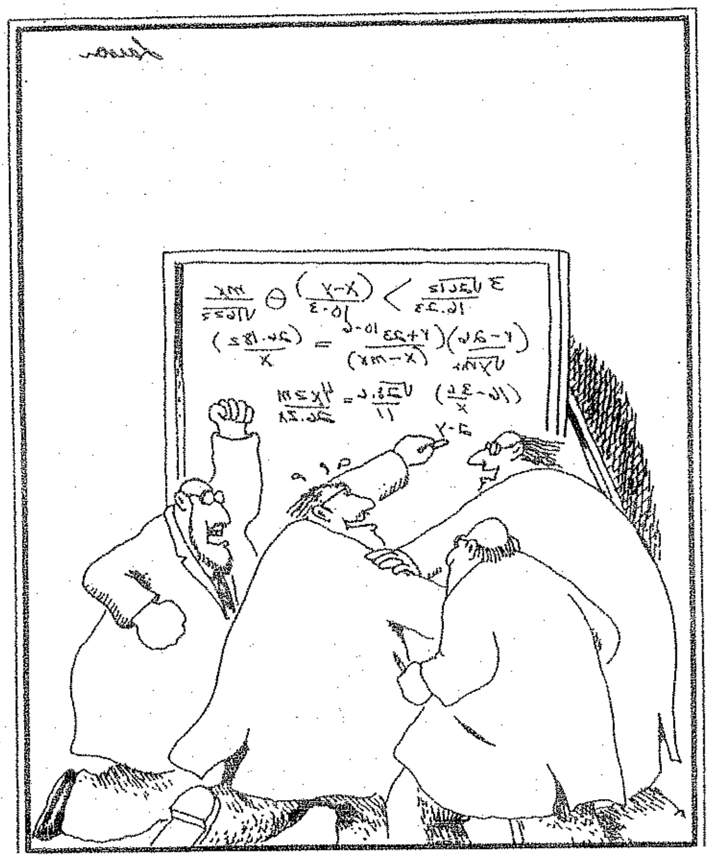
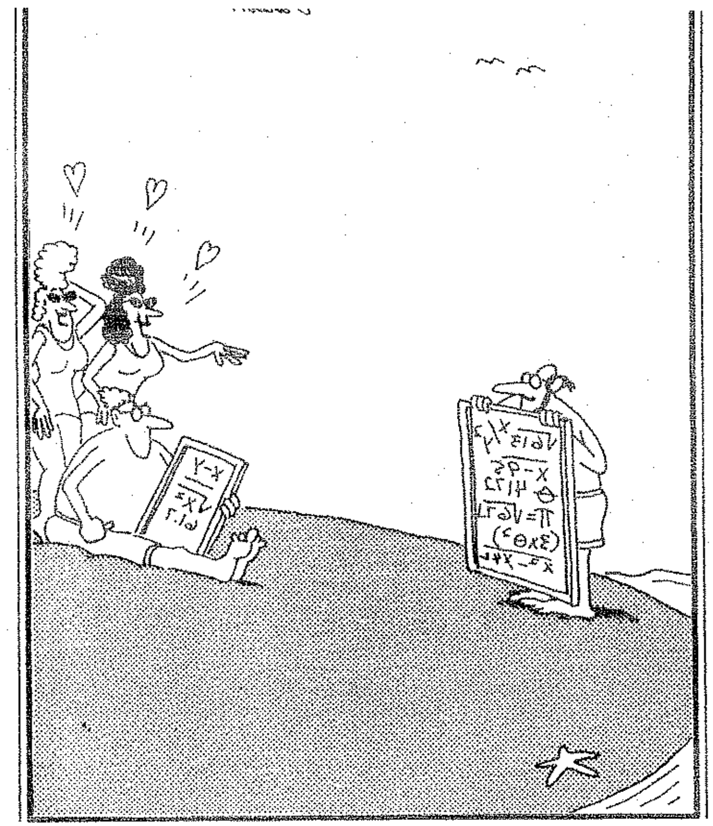


Figure 29.11. Rotation curve (Schmidt model) for our Galaxy, in km sec^{-1} kpc^{-1}, and epicyclic frequency.



"Go for it, Sidney! You've got it! You've got it! Good hands! Don't choke!"



Spiral Density Waves

Lets consider small spiral perturbations within an otherwise axi-symmetric disk. Assume further that the pattern is pre-existing and we will develop a description of the response.

We must develop densities, the potential and the response self consistently. Note: typical contrast is ~10% (in light) so small perturbation (1st order) is sufficient.

Consider a thin disk with scales:

$$Z/R \sim \frac{0.3\ \text{kpc}}{15\ \text{kpc}} \sim 0.02$$

– only $\sigma(r)$

Velocity field will be circular:

$$V_\theta = \Omega(r)\, r\,, \quad \sigma = \sigma(r)$$

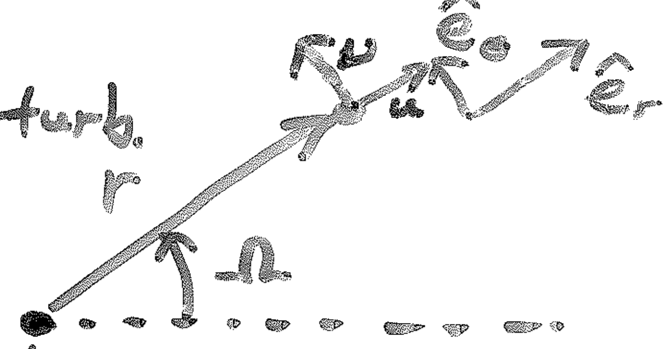
However, in the perturbed state:

Velocity:

ang. perturb.

$$V(r,\theta) = (u, \nu + \underbrace{r\Omega}_{\text{circ. component.}})$$

radial perturb.



Density:

$$\sigma(r,\theta,t) = \sigma_0(r) + \sigma'(r,\theta,t)$$

axisymmetric

The motion of a gas element is described by the hydrodynamic equations in cylindrical coordinates:

① Continuity equation (conserv. of mass)

$$\nabla \cdot (\rho \hat{V}) = -\frac{\partial \rho}{\partial t}$$

where $\rho = \sigma(r,\theta)$ and ② $\hat{V} = u\hat{e}_r + (\nu + r\Omega)\hat{e}_\theta$

In cylindrical coords:

$$\nabla = \hat{e}_r \, \partial/\partial r + \left(\frac{\hat{e}_\theta}{r}\right)(\partial/\partial\theta)$$

(recall $\theta = s/r$, $\hat{e}_\theta = \hat{e}_r/r$)

so: $$\nabla \cdot (\sigma \hat{V}) = \hat{V} \cdot \nabla \sigma + \sigma \nabla \cdot \hat{V}$$

evaluating each term:

From ② the first term becomes:

$$\hat{V}\cdot\nabla\sigma = \underbrace{u\frac{\partial \sigma}{\partial r}}_{r} + \underbrace{(v+r\Omega)\frac{1}{r}\frac{\partial \sigma}{\partial \theta}}_{\theta}$$

and the second term becomes:

$$\sigma\nabla\cdot\hat{V} = \sigma\frac{\partial u}{\partial r} + \frac{\sigma}{r}\frac{\partial}{\partial \theta}(v+r\Omega) + \frac{u\sigma}{r}$$

where the last term is due to $\frac{1}{r}\frac{\partial \hat{e}_r}{\partial \theta}$.

The continuity equation then becomes:

$$① \quad \frac{\partial \sigma}{\partial t} + \underbrace{\frac{1}{r}\frac{\partial}{\partial r}(ru\sigma)}_{r} + \underbrace{\frac{1}{r}\frac{\partial}{\partial \theta}\sigma(v+r\Omega)}_{\theta} = 0$$

Note: This is the time rate of change of the mass ($\sigma r \Delta r \Delta\theta$) in the volume element ($r\Delta r\Delta\theta$) in the radial (2nd term) and angular (3rd term) directions.

Now consider the equation of motion (momentum equation: Lagrangian form with scalar P):

$$\rho\frac{d\hat{V}}{dt} = \hat{g} - \nabla P, \text{ where } \frac{d\hat{V}}{dt} = \frac{\partial \hat{V}}{\partial t} + \hat{V}\cdot\nabla\hat{V}$$

Note that $\frac{\partial \hat{e}_r}{\partial \theta} = \hat{e}_\theta$ and $\frac{\partial \hat{e}_\theta}{\partial \theta} = -\hat{e}_r$

SO:

$$\hat{V}\cdot\nabla\hat{V} = u\frac{\partial\hat{V}}{\partial r} + \frac{(v+r\Omega)}{r}\frac{\partial\hat{V}}{\partial\theta}$$

$$= u\frac{\partial u}{\partial r}\hat{e}_r + u\frac{\partial}{\partial r}(v+r\Omega)\hat{e}_\theta \quad \text{1st term}$$

$$+\frac{(v+r\Omega)}{r}\frac{\partial u}{\partial\theta}\hat{e}_r + \frac{u(v+r\Omega)}{r}\underbrace{\hat{e}_\theta}_{\frac{\partial\hat{e}_r}{\partial\theta}}$$

$$-\frac{(v+r\Omega)^2}{r}\underbrace{\hat{e}_r}_{\frac{\partial\hat{e}_r}{\partial\theta}} + \hat{e}_\theta\frac{(v+r\Omega)}{r}\frac{\partial}{\partial\theta}(v+r\Omega)$$

Now consider the individual terms:

② radial

$$\sigma\left\{\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{(v+r\Omega)}{r}\frac{\partial u}{\partial\theta} - \frac{(v+r\Omega)^2}{r}\right\} = g_r - \frac{\partial P}{\partial r}$$

③ angular

$$\sigma\left\{\frac{\partial v}{\partial t} + u\frac{\partial}{\partial r}(v+r\Omega) + u\frac{(v+r\Omega)}{r} + \frac{(v+r\Omega)}{r}\frac{\partial v}{\partial\theta}\right\} = g_\theta - \frac{1}{r}\frac{\partial P}{\partial\theta}$$

$r\Omega$ is independ.
of θ and t

where $g_r = -\sigma \frac{\partial \Phi}{\partial r}$ and $g_\theta = \frac{-\sigma}{r} \frac{\partial \Phi}{\partial \theta}$

and Φ is the potential.

If the gas is in turbulent motion the pressure across a surface is related to the momentum density (σa_0) and the turbulent speed (a_0). That is, since $P = \rho v^2$ we have:

$$\frac{\partial P}{\partial r} = \frac{\partial}{\partial r} \sigma a_0^2 = a_0^2 \frac{\partial \sigma}{\partial r} \quad \text{and}$$

$$\frac{\partial P}{\partial \theta} = \frac{\partial}{\partial \theta} \sigma a_0^2 = a_0^2 \frac{\partial \sigma}{\partial \theta}$$

pressure gradient $\Longleftrightarrow$ density gradient

Hydrodynamic Equation relate variables of the gas (u, v, σ) to the potential Φ, which we obtain from <u>Poisson's equ</u>:

$$\nabla^2 \Phi = 4\pi G \sigma \delta(z)$$

This can be solved numerically (e.g. Aoki et al. 1979 PASJ 31, 737).

However if we assume the primary

response will result from local perturbations (similar to WKB approx. in quantum). By making some approximations we can obtain analytic solutions.

We begin by "linearizing" the hydrodymic equation (i.e. we consider only 1st order terms) and the equations of motion in the case of small perturbations to Φ, σ. Specifically, let the potential be given by:

$$\Phi(r,\theta,t) = \Phi_0(r,z) + \Phi'(r,\theta,z,t)$$

↑ no time dependence (axisymm. disk model)

and the density be given by:

$$\sigma(r,\theta,t) = \sigma_0(r) + \sigma'(r,\theta,t)$$

↑ no time dependence

Substituting σ into the continuity equation (1) will give the contin. equ. for 1st order pertubations:

 so:

$$\underline{\frac{\partial \sigma_0}{\partial t}} + \frac{\partial \sigma'}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r u \sigma_0) + \underline{1/r\frac{\partial}{\partial r}(r u \sigma')}$$

$$+ 1/r \frac{\partial}{\partial \theta}\sigma_0(\nu + \underline{r\Omega}) + 1/r\frac{\partial}{\partial \theta}\sigma'(\underline{\nu} + r\Omega) = 0$$

Note $\frac{\partial \sigma_0}{\partial t} = 0$ (no time dependence)

and $1/r\frac{\partial}{\partial \theta}(\sigma_0 r \Omega) = 0$ (no θ dependence)

Note also we will drop 2nd order terms

$1/r\frac{\partial}{\partial r}(r\underline{u\sigma'})$ and $1/r\frac{\partial}{\partial \theta}(\underline{\nu\sigma'})$ since these are the product of two small quantities and are thus 2nd order.

The resulting continuity equation to 1st order becomes:

$$④\quad \frac{\partial \sigma'}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r u \sigma_0) + \frac{\sigma_0}{r}\frac{\partial \nu}{\partial \theta} + \Omega\frac{\partial \sigma'}{\partial \theta} = 0$$

Following a similar procedure we can derive the equations of motion for 1st order perturbations. But first lets rewrite the radial equation using the contin. equation. That is: $\sigma \frac{\partial u}{\partial t} = \frac{\partial(\sigma u)}{\partial t}$ and $\sigma u \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\sigma u^2)$ so we have:

$$(2')\quad \frac{\partial}{\partial t}(\sigma u) + \frac{1}{r}\frac{\partial}{\partial r}(r\sigma u^2) + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\sigma u(v+r\Omega)\right] - \frac{\sigma}{r}(v+r\Omega)^2 = -\frac{\partial P}{\partial r} - \sigma\frac{\partial\Phi}{\partial r}$$

now substituting for σ and Φ we get:

$$\sigma_0 \frac{\partial u}{\partial t} + \cancelto{0}{\frac{\partial(\sigma' u)}{\partial t}} + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial r}(r\sigma u^2)} + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}(\sigma_0 uv)} +$$

$$\frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}(\sigma' uv)} + \frac{1}{r}\frac{\partial}{\partial\theta}(\sigma_0 u r\Omega) + \frac{1}{r}\cancelto{0}{\frac{\partial}{\partial\theta}(\sigma' u r\Omega)}$$

$$-\frac{\sigma}{r}\left[\cancelto{0}{v^2} + 2vr\Omega + \underline{r^2\Omega^2}\right] = -a_0^2\left(\underline{\frac{\partial\sigma_0}{\partial r}} + \frac{\partial\sigma'}{\partial r}\right) -$$

$$\sigma_0\left(\underline{\frac{\partial\Phi}{\partial r}} + \frac{\partial\Phi'}{\partial r}\right) - \sigma'\left(\underline{\frac{\partial\Phi}{\partial r}} + \cancelto{0}{\frac{\partial\Phi'}{\partial r}}\right)$$

__Note:__ the underlined terms can be collected. We find:

$$-\sigma r\Omega^2 = -a_0^2\frac{\partial\sigma_0}{\partial r} - \sigma\frac{\partial\Phi}{\partial r}$$ or:

$$-\frac{\partial\Phi}{\partial r} = -r\Omega^2 + \frac{a_0^2}{\sigma}\frac{\partial\sigma_0}{\partial r}$$ (grad. of potential = net force)

$$= \underbrace{-r\Omega^2}_{\text{centripital force}} + \underbrace{\frac{1}{\sigma}\frac{\partial P}{\partial r}}_{\text{pressure grad.}}$$

So these terms cancel each other.

Dividing through by σ_0 gives:

1st order radial

(5) $$\frac{\partial u}{\partial t} + \Omega \frac{\partial u}{\partial \theta} - 2 v \Omega = \frac{-a_0^2}{\sigma_0} \frac{\partial \sigma'}{\partial r} - \frac{\partial \Phi'}{\partial r}$$

(leaving only perturbed terms)

Following a similar procedure for the azimuthal equation (3) (subsituting for σ and Φ) gives:

$$\sigma_0 \frac{\partial v}{\partial t} + \underline{\sigma' \frac{\partial v}{\partial t}} + \underline{\sigma_0 u \frac{\partial v}{\partial r}} + \underline{\sigma' u \frac{\partial v}{\partial r}} + \sigma_0 u \frac{\partial (r\Omega)}{\partial r}$$

$$+ \underline{\sigma' u \frac{\partial (r\Omega)}{\partial r}} + \underline{\frac{\sigma_0 u v}{r}} + \underline{\frac{\sigma' u v}{r}} + \sigma_0 u \Omega + \underline{\sigma' u \Omega}$$

$$+ \underline{\frac{\sigma_0 v}{r} \frac{\partial v}{\partial \theta}} + \underline{\frac{\sigma' v}{r} \frac{\partial v}{\partial \theta}} + \sigma_0 \Omega \frac{\partial v}{\partial \theta} + \underline{\sigma' \Omega \frac{\partial v}{\partial \theta}}$$

$$= \frac{-1}{r} \frac{\partial \Phi}{\partial r} (\sigma_0 + \sigma') - \frac{1}{r} \left[a_0^2 \left(\cancelto{0}{\frac{\partial \sigma_0}{\partial \theta}} + \frac{\partial \sigma'}{\partial \theta} \right) \right]$$

<u>Note</u> the underlined terms are 2nd order (or small) and dropped. So we have:

$$\sigma_0 \frac{\partial v}{\partial t} + \sigma_0 u \frac{\partial (r\Omega)}{\partial r} + \sigma_0 u \Omega + \sigma_0 \Omega \frac{\partial v}{\partial \theta} =$$

$$- 1/r \left(a_0^2 \frac{\partial \sigma'}{\partial \theta} + \frac{\partial \Phi}{\partial \theta} \right)$$ or:

1st order angular

(6) $$\frac{\partial v}{\partial t} + u \frac{\partial (r\Omega)}{\partial r} + u\Omega + \Omega \frac{\partial v}{\partial \theta} = \frac{-1}{r} \left(\frac{a_0^2}{\sigma_0} \frac{\partial \sigma'}{\partial \theta} + \frac{\partial \Phi}{\partial \theta} \right)$$

expand?

And for Poisson's equation we get:

$$\nabla^2 \Phi' = 4\pi G \sigma' \delta(z) \quad (7)$$

We now have all the tools we need to develop solutions to the hydrodynamic equations for spiral perturbations. However lets first catch our breath and discuss epicyclic motion in more detail.

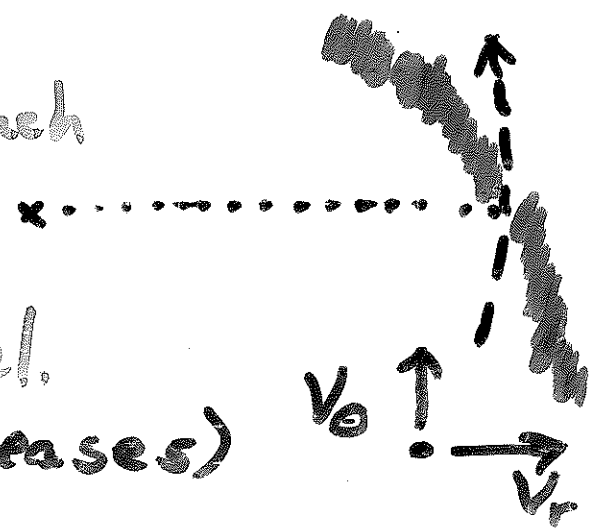
Epicyclic Motion Continued

~2/3 of luminous galaxies show spiral structure. Recall the "winding problem". We need the spiral pattern (Ω_p) to be ~ independent of radius. However stars and gas show a flat rotation curve: $V_{max} \sim$ constant

so $\Omega_* = V_{max}\, r \neq$ constant

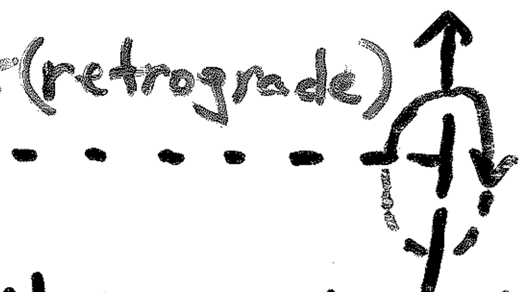
Consider a low amplitude (~10%) perturbing potential.

1) As particles approach spiral arm both V_r and V_θ increase due to outward accel. **(orbital period increases)**



2) As particles leave spiral they feel inward accel. and they slow down.

3) In the rotating frame the result is an epicycle about the mean circular orbit (retrograde)



Let's look at this mathematically.

Let ξ and η be the perturbations in the r and θ directions, respectively. In the radial direction we can write Newton's 2^{nd} law as:

$\ddot{r} = F_r + r\dot{\theta}^2$ where $F_r = \frac{-V^2}{r_0}$ is the centripital force/mass and $\dot{\theta}$ is the angular velocity V_θ/r. Obviously, if there is no net radial force $\ddot{r} = 0$ and since $V_{\theta,c} = \dot{\theta}/r_0$ then $r = r_0$.

For small perturbations let:

$$r = r_0 + \xi, \quad \dot{r} = \dot{\xi}, \quad \ddot{r} = \ddot{\xi}$$

The net force then becomes:

$$① \quad \ddot{\xi} = -\frac{V_c^2}{r} + \frac{V_\theta^2}{r}$$

Expanding r^{-1} as a Taylor series:

$$r^{-1} = \frac{1}{r_0 + \xi}$$

$$= \frac{1 - \xi/r_0}{(r_0+\xi)(1-\xi/r_0)}$$

$$= \frac{1 - \xi/r_0}{r_0 - \xi + \xi - \xi^2/r_0}$$

$$② \quad r^{-1} \simeq (1 - \xi/r_0)/r_0 \quad \text{since } \xi^2/r_0 \simeq 0$$

Similarly:

$$③ \quad V_c(r) = V_c(r_0 + \xi) \simeq V_c(r_0) + \left(\frac{dV_c}{dr}\right)_{r=r_0} \xi$$

Conservation of angular momentum gives

$$V_\theta(r)\, r = V_c(r_0)\, r_0 \quad \text{so:}$$

$$④ \quad \frac{V_\theta(r)}{r} = \frac{V_c(r_0)\, r_0}{r^2} \simeq \frac{V_c(r_0)}{r_0}\left[1 - 2(\xi/r_0)\right]$$

(substituting ② and dropping ξ^2 terms)

Substituting 3,4 into 1 and letting $V_0 = V_c(r_0)$

$$⑤ \quad \ddot{\xi} \simeq \frac{(1 - \xi/r_0)}{r_0}\left\{\frac{r_0^2 V_0^2}{r^2} - \left(V_0 + \xi \frac{dV_0}{dr}\right)^2\right\}$$

(after dropping terms of order ξ^2)

Now we expand ξ to 1st order:

From 4: $\dfrac{V_0 r_0}{r^2} \simeq \dfrac{V_0}{r_0}\left[1 - 2(\xi/r_0)\right]$ so:

$$\ddot{\xi} = \frac{(1 - \xi/r_0)}{r_0}\left\{V_0^2\left[1 - 2(\xi/r_0)\right] - \left[V_0^2 + 2V_0\xi\frac{dV}{dr} + \cancel{\xi^2\left(\frac{dV}{dr}\right)^2}^{\,0}\right]\right\}$$

$$= \frac{(1 - \xi/r_0)}{r_0}\left\{-2V_0^2(\xi/r_0) - 2V_0\,\xi\,\frac{dV}{dr}\right\}$$

$$= \frac{-2V_0^2 \xi}{r_0^2} + \cancel{\frac{2V_0^2 (\xi/r_0)^2}{r_0}} - \frac{2V_0 \xi \left(\frac{dV}{dr}\right)}{r_0}$$

$$- \cancel{\frac{2V_0 \xi^2 \frac{dV}{dr}}{r_0^2}}$$

so:

$$\ddot{\xi} = \frac{-2V_0^2}{r_0^2} \xi \left[1 + \frac{r_0}{V_0} \left(\frac{dV}{dr}\right)_{r_0}\right]$$

or we can write:

$$\ddot{\xi} = -K^2 \xi \quad \text{where } K^2 = \frac{2V_0^2}{r_0^2}\left[1 + \frac{r_0}{V_0}\left(\frac{dV}{dr}\right)_{r_0}\right]$$

This is the equation for simple harmonic motion! That is, motion about r_0 with frequency $2\pi/K$.

For initial conditions $t_0 = 0$, $\xi(0) = 0$, and $\dot{\xi}(0) = V_r(0)$ then:

$$(7) \quad \xi(t) = \frac{V_r(0)}{K} \sin(Kt)$$

Note that we can rewrite K as

$$K^2 = \Omega^2 \left[1 + \frac{r_0}{2\Omega}\left(\frac{d\Omega}{dr}\right)_{r_0}\right] \text{ since}$$

$r\Omega = V_c(r)$ at $r = r_0$

<u>Note</u> if the rotation curve is flat

$\frac{dV_c}{dr} = 0$ then $K = \sqrt{2}\, V_0/r_0$ and

$$P_{rad} = \frac{2\pi}{\sqrt{2}} \frac{r_0}{V_0}$$

Since the orbital period is:

$$P_0 = \frac{2\pi r_0}{V_0}$$

$$\frac{P_{rad}}{P_0} = \frac{2\pi r_0}{\sqrt{2}\, V_0} \cdot \frac{V_0}{2\pi r_0} = \frac{1}{\sqrt{2}} \sim 71\%$$

So orbits are not closed and precess.

Now consider perturbations in the angular direction (z).

Conservation of angular momentum:

$$rV_\theta = r_0 V_0 = r^2 \Omega = r^2 \frac{d\theta}{dt}$$

so:

$$\dot\theta = \frac{r_0 V_0}{r^2} \simeq \underbrace{\frac{V_0}{r_0}}_{\dot\theta_0 = \Omega_0} - \frac{2V_0}{r_0^2}\,\xi \quad \text{from } ④$$

and:

$$\dot\theta - \dot\theta_0 = \frac{-2V_0}{r_0^2}\,\underbrace{\xi = \frac{-2V_0 V_r}{r_0^2 K} \sin(Kt)}_{\text{from } 7}$$

multiplying through by r_0 we have:

$$\underbrace{r_0\dot\theta - r_0\dot\theta_0}_{-\Delta V_\theta \sim -\dot\eta} = \frac{-2V_0 V_r}{r_0 K} \sin(Kt)$$

Integrating we obtain:

$$\eta(t) = \frac{2V_0 V_r}{r_0 K} \cos(Kt)$$

(epicyclic motion with the same period)

The ratio of amplitudes gives axial ratio of the epicycle:

$$\frac{\eta}{\xi} = \frac{2V_0}{r_0 K}$$ but for flat rotation curves $K \simeq \sqrt{2}\, V_0/r_0$

So the azimuthal amplitude is $\sim\sqrt{2}$ times the radial amplitude.

$$\eta_{max} \simeq \sqrt{2}\ \xi_{max}$$

Now lets consider Spiral pertubations. The primary references are:

Lin & Shu 1964 ApJ 140, 646

Toomre 1977 ARAA 15, 437

Binney & Tremaine Ch. 6 (Galactic Dyn.)

Geometric spirals follow the general form $n\theta = \varphi(r)$ where $\varphi(r)$ is a monotonically increasing function. Examples include:

Spiral of Archimedes:

$$r = a\theta$$

Logarithmic Spiral:

$$r = e^{a\theta}$$

Since $\tan\alpha = n / \left(r \frac{d\psi}{dr}\right)$ we can define a wave vector k (not K!) as

$$k = \left[\frac{d\psi}{dr}\right]$$

so:

$$\tan\alpha = \frac{n}{kr}$$

with a corresponding wavelength (λ):

$10^\circ \lesssim \alpha \lesssim 20^\circ$

$$n\Delta\theta = \psi(r+\lambda) - \psi(r) = 2\pi n$$

so if λ/r is small $\psi(r+\lambda) \simeq \psi(r) + k\lambda$

and $\lambda = \frac{2\pi n}{k}$

Let's assume we express the perturbati as a power series of modes (n). So we have some set of functions F.

$$F(r,\theta,t) = \sum_n f_n(r)\, e^{i(\omega t - n\theta)}$$

(log. spiral)

where n is an integer ≥ 1. We expect $F(r,\theta,t)$ to have "spiral properties". Also: the pattern must be similar for rotations of $\Delta\theta \simeq 2\pi/n$ so $n\theta - \psi(r) \sim$ constant

Thus we will consider the following forms for the perturbations:

$$\sigma_n' = \tilde{\sigma}_n \exp i\left[\omega t - n\theta + \Psi(r)\right]$$
$$u_n = \tilde{u}_n \exp i\left[\omega t - n\theta + \Psi(r)\right]$$
$$v_n = \tilde{v}_n \exp i\left[\omega t - n\theta + \Psi(r)\right]$$
$$\Phi_n' = \tilde{\Phi}_n \exp i\left[\omega t - n\theta + \Psi(r)\right]$$

where $\Psi(r)$ is the "form function" of the spiral.

Note: for a spiral with constant α:

check this $\Big\{$ $k = \dfrac{k_0}{r}$ and $\tan\alpha = n/kr$ but
$k = \dfrac{d\Psi}{dr}$ and $\tan\alpha = \dfrac{dr}{d\theta}\,1/r$ so:

$$\frac{d\Psi}{dr} = C\,n\,\frac{d\theta}{dr} = \frac{k_0}{r} \quad \text{so:}$$

$$k_0 \frac{dr}{r} = C\,n\,d\theta$$

Integrating:

$$k_0 \ln(r) = C n \theta \quad \text{so} \quad r(\theta) = C_0 e^{n/k_0\,\theta}$$

a logarithmic spiral! α = constant

Solutions to the 1st-order theory can be found by substituting the perturbations into the continuity equation and the equations of motion. Begin with the contin equn:

$$\frac{\partial \sigma'}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(ru\sigma_0) + \frac{\sigma_0}{r}\frac{\partial v}{\partial \theta} + \Omega\frac{\partial \sigma'}{\partial \theta} = 0$$

<u>Now</u> $\frac{\partial \sigma'}{\partial t} = (i\omega)\sigma_n \exp i[\omega t - n\theta + \psi(r)]$
$= i\omega\sigma'$

$$\frac{\partial v}{\partial \theta} = -inv \text{ and } \frac{\partial \sigma'}{\partial \theta} = -in\sigma'$$

$$\frac{\partial}{\partial r}(ru\sigma_0) = \sigma_0\Big(u + \underbrace{iru\frac{\partial \psi}{\partial r} + \frac{u}{\tilde{u}}\frac{\partial \tilde{u}}{\partial r}}_{\text{chain rule}}\Big)$$

Grouping terms we have:

$$i\omega\sigma' + \frac{\sigma_0}{r}\left(u + iru\frac{\partial \psi}{\partial r} + \frac{u}{\tilde{u}}\frac{\partial \tilde{u}}{\partial r}\right)$$
$$-\frac{\sigma_0}{r}inv - \Omega in\sigma' = 0$$

since all the terms contain $\exp i[\omega t - n\theta + \psi(r)]$ we have

$$i\omega\tilde{\sigma} + \frac{\sigma_0}{r}\left(\tilde{u} + ir\tilde{u}\frac{\partial\psi}{\partial r} + \frac{\partial\tilde{u}}{\partial r}\right)$$
$$- \frac{in\sigma_0\tilde{v}}{r} - in\tilde{\sigma}\Omega = 0$$

Since $k = \frac{\partial\psi}{\partial r}$ we have:

$$i\omega\tilde{\sigma} + \frac{\sigma_0\tilde{u}}{r} + i\sigma_0\tilde{u}k + \frac{\sigma_0}{r}\frac{\partial\tilde{u}}{\partial r} - \frac{in\sigma_0\tilde{v}}{r}$$
$$- in\tilde{\sigma}\Omega = 0$$

grouping by $\tilde{\sigma}, \tilde{u}, \tilde{v}$ we have:

$$i\tilde{\sigma}(\omega - n\Omega) + k\sigma_0\tilde{u}\left(i + \frac{1}{kr}\right)$$
$$= \sigma_0 k\, \tilde{v}/kr \; in$$

dividing by iK (not k) gives:

$$\frac{\tilde{\sigma}(\omega - n\Omega)}{K} + \frac{k\sigma_0\tilde{u}}{K}\left(1 - \frac{i}{kr}\right) = \frac{\sigma_0 k}{K}\frac{\tilde{v}}{kr}\, n$$

Taking the time derivative of the density wave Φ' (i.e. the [ ]):

$$\frac{d}{dt}\left[\omega t - n\theta + \psi(r)\right] = \omega - n\frac{d\theta}{dt}$$

But since the wave has pattern speed of $r\Omega_p$ where $\frac{d\theta}{dt} = \Omega_p$ then $\omega/n \equiv \Omega_p$ so:

$$\frac{n(\Omega_p-\Omega)\tilde{\sigma}}{K}+\frac{k\sigma_0}{K}\left(1-\cancel{\frac{i}{kr}}^{\,0}\right)\tilde{u}=-\frac{\sigma_0 k}{K}\cancel{\frac{\tilde{\sigma}n}{kr}}^{\,0}$$

Since $\tan\alpha=\frac{n}{kr}$ and $\alpha\sim 15^\circ$ then $kr\gg n$ and since $n\sim 2$ $1/kr$ is ~~small too~~. The result is:

$$\text{(1)}\qquad \frac{n(\Omega_p-\Omega)\tilde{\sigma}}{K}+\frac{k\sigma_0\tilde{u}}{K}=0$$

Proceeding with the equations of motion:

radial
$$\frac{\partial u}{\partial t}+\Omega\frac{\partial u}{\partial\theta}-2v\Omega=\frac{-a_0^2}{\sigma_0}\frac{\partial\sigma'}{\partial r}-\frac{\partial\Phi'}{\partial r}$$

$\frac{\partial u}{\partial t}=i\omega u$, $\frac{\partial u}{\partial\theta}=-inu$ and

$\frac{\partial\sigma'}{\partial r}=i\tilde{\sigma}\frac{\partial\psi}{\partial r}+\frac{1}{\tilde{\sigma}}\frac{\partial\tilde{\sigma}}{\partial r}$ so we have

$$i\omega u-\Omega inu-2v\Omega=\frac{-a_0^2}{\sigma_0}\left(i\tilde{\sigma}k+\cancel{\frac{1}{\tilde{\sigma}}\frac{\partial\tilde{\sigma}}{\partial r}}^{\,0}\right)-\frac{\partial\Phi'}{\partial r}$$

23 Grouping terms and dividing by K:

$$\frac{iu(\omega - m\Omega)}{K} - \frac{2v\Omega}{K} = \frac{-ia_0^2 k\sigma'}{\sigma_0 K} - \frac{\partial \Phi'}{K\partial r}$$

So:

$$\frac{iu(\Omega_p - \Omega)n}{K} + \frac{ia_0^2 k\sigma'}{\sigma_0 K} - \frac{2v\Omega}{K} = \frac{g_r}{\sigma' K}$$

since all terms contain

$\exp i[\omega t - n\theta + \psi(r)$ we have:
(amplitudes only)

$$\frac{i\tilde{u}(\Omega_p - \Omega)n}{K} + \frac{ia_0^2 k\tilde{\sigma}}{\sigma_0 K} - \frac{2\tilde{v}\Omega}{K} = \frac{g_r}{K}$$

something wrong here

② $$\frac{ia_0^2 k}{\sigma_0 K} + if_* - \frac{2\Omega\tilde{v}}{K} = g_r/K$$

where $f_* \equiv \dfrac{n(\Omega_p - \Omega)}{K}$

<u>azimuthal</u>

$$\frac{\partial v}{\partial t} + u\frac{\partial(r\Omega)}{\partial r} + u\Omega + \Omega\frac{\partial v}{\partial \theta} = -\frac{1}{r}\left(\frac{a_0^2}{\sigma_0}\frac{\partial \sigma'}{\partial \theta} + \frac{\partial \Phi}{\partial \theta}\right)$$

where

24

$$\frac{\partial \nu}{\partial t} = i\omega\nu \ , \ \frac{\partial \nu}{\partial \theta} = -in\nu \ , \ \frac{\partial \sigma'}{\partial \theta} = -in\sigma'$$

and

$$\frac{\partial (r\Omega)}{\partial r} = \Omega$$

so

$$i\omega\nu + u\Omega + u\Omega - in\nu\Omega = -\frac{1}{r}\left(\frac{a_0^2 in\sigma'}{\sigma_0} + \cancel{\frac{\partial \Phi}{\partial \theta}}\right)$$

0

some problem here

$$i\nu(\omega - n\Omega) + 2u\Omega = 0$$

$$\frac{i\nu n(\Omega_p - \Omega)}{K} + \frac{2u\Omega}{K} = 0$$

$$③ \quad i\tilde{\nu} f_* + \frac{2\Omega}{K}\tilde{u} = 0$$

(amplitude only)

so we have:

$$④ \begin{cases} \tilde{\sigma}/\sigma_0 = \dfrac{-ikg_r}{K^2(1-f_*^2)+k^2a_0^2} \\ \tilde{u} = \dfrac{iKg_r f_*}{K^2(1-f_*^2)+k^2a_0^2} \\ \tilde{\nu} = \dfrac{K^2 g_r/2\Omega}{K^2(1-f_*^2)+k^2a_0^2} \end{cases}$$

Poisson's Eqn. relates σ to g_r. This gives 4 eqn. and 4 unknowns.

Recall that $g_r = 2\pi i G\sigma \, K/|K|$

So from the first equation:

$$\frac{|k|}{k_0} = (1-f_*) + \frac{k^2 a_0^2}{K} \quad \text{where}$$

(dispersion relation) $\quad k_0 = \frac{K^2}{2\pi G \sigma_0}$

So

$$|k| = \frac{K^2}{\pi G \sigma_0}\left[1 \pm n(\Omega - \Omega_p)/K\right]$$

If we require the wave # be real:

$$\frac{4 k^2 a_0^2}{K^2}(1-f_*) \leq 1 \qquad f_* \geq 0$$

so the stabillity condition becomes:

$f_* = 0$ so we have

$$a_0 < \frac{K}{2k_0} \equiv \frac{\pi G \sigma_0}{K}$$

and stable spiral structure demands:

$$a_0 < \frac{\pi G \sigma_0}{K}$$

If turbulent velocity is too high no spiral struct.

Note further that since

$$\frac{4k_0^2 a_0^2}{K^2}(1-f_*^2) \leq 1 \quad \text{then}$$

$$1-f_*^2 \geq 0 \quad \text{since } f_*^2 = \frac{n^2(\Omega_p-\Omega)^2}{K^2}$$

$$1-\frac{n^2(\Omega_p-\Omega)^2}{K^2} \geq 0$$

so

$$\left(\frac{K}{n}\right)^2 > (\Omega_p-\Omega)^2$$

$$\left|\Omega \pm \frac{K}{n}\right| > \Omega_p$$

$$* \quad \Omega - \frac{K}{n} \leq \Omega_p \leq \Omega + \frac{K}{n}$$

Note that this is a resonance condition ($\Omega - \Omega_p = \pm K/n$). In this case a star encounters the density wave at the same point (phase) in its epicyclic orbit. These are the Linblad resonances.

Spiral Structure should only exist between the ILR and the OLR.

Co-Rotation - $\Omega = \Omega_p$ so
a strong resonance

- location of strong rings of star formation

- Bars located within CR

<u>Note</u> ILR often associated with "nuclear rings" of star formation.

Comparison with Observations

What do we expect?

1) "large" velocity dispersions for gas and stars located in spiral arms

2) peculiar (i.e. noncircular) velocities for gas near spiral arms

<u>Specifically</u>

$$V(r) = r\Omega(r) + \nu$$
$$= \underbrace{r\Omega(r)}_{\text{circular part}} + \underbrace{\tilde{\nu}\cos(\Lambda \ln r/r_0)}_{\text{perturbed velocity}}$$

<u>Recall</u>

$$\nu = \tilde{\nu} \exp i[\omega t - n\theta - \psi(r)]$$

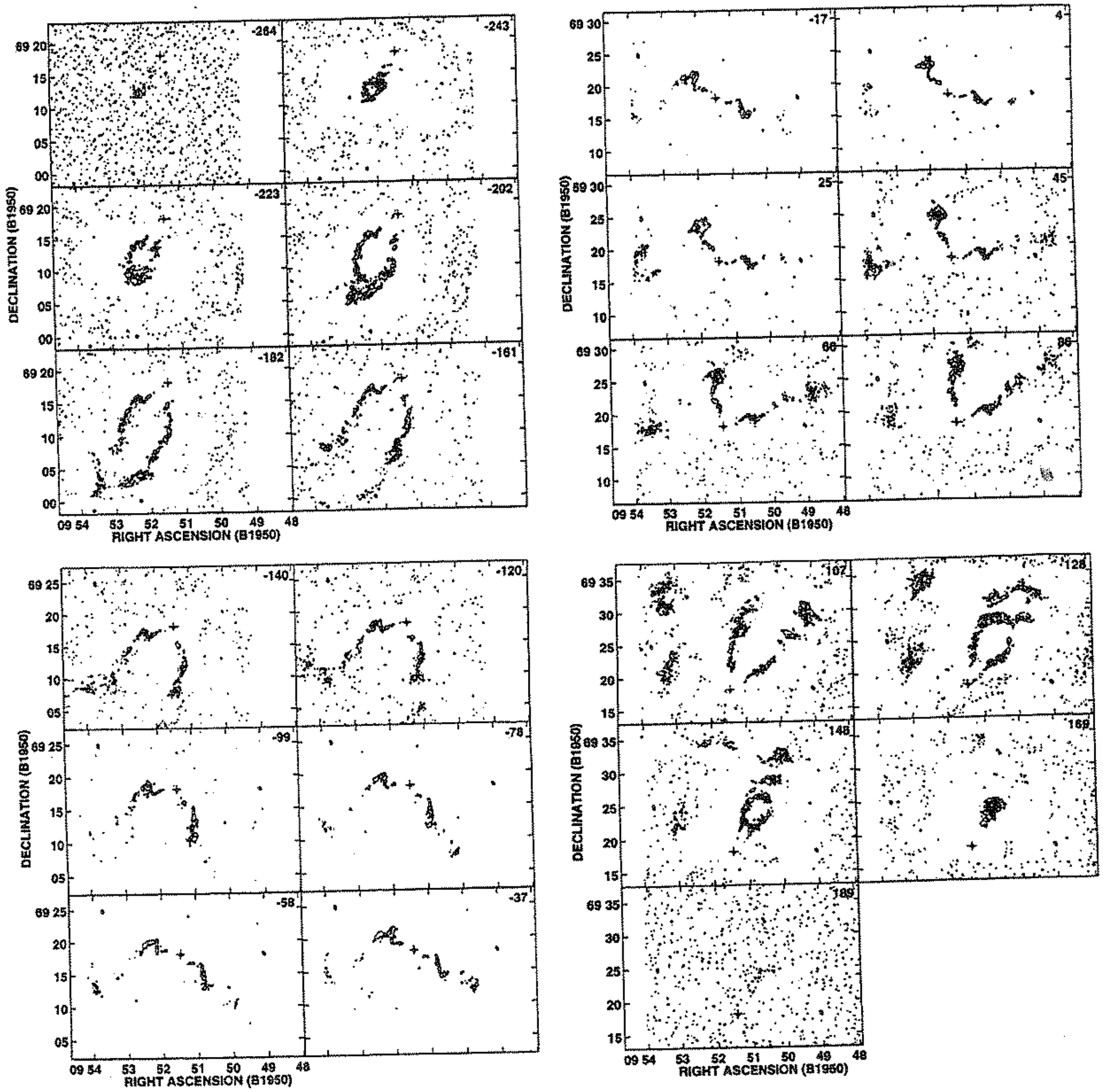
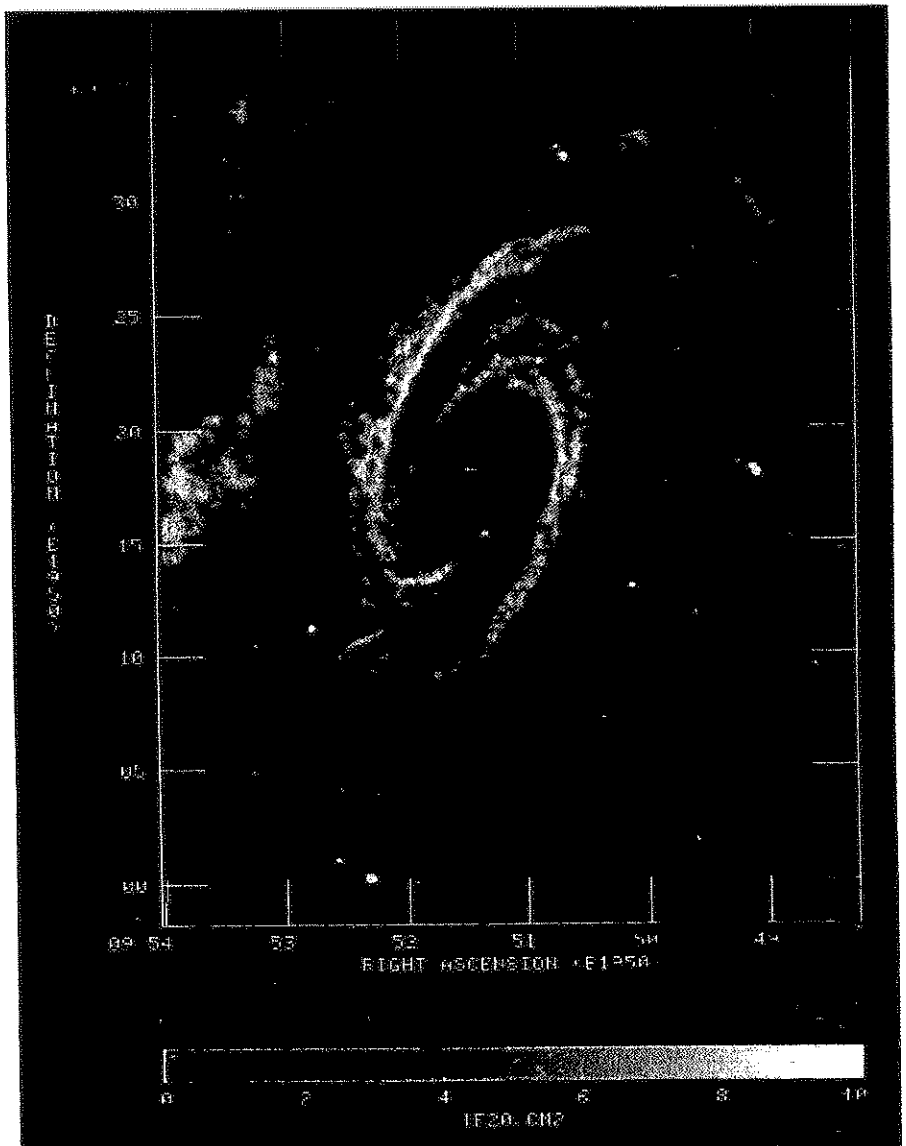


FIG. 2. Channel maps for the full-resolution (not primary beam corrected) data set. Every eighth channel is displayed; the corresponding optical barycentric velocity is displayed in the upper right of each panel. The beamsize in this (and all subsequent images) is 12″×12″. Contour levels correspond to 1.5, 5, and 10 mJy beam^{-1}; the rms noise per channel is on the order of 0.8 mJy beam^{-1}. A cross marks the center of the galaxy. The continuum has not been subtracted from these maps (see text).

HI Velocity Field of M81

Adler & Westpfahl 1996 AJ 111, 735



. 4. Integrated H I intensity map, calculated from −277 to 195 km s^{-1}. The pixel size in this image is 5″. The intensity scale is in units of 10^{20} atoms cm^{-2}. e estimate the rms noise in this image to be 1×10^{19} atoms cm^{-2}. A cross marks the center of the galaxy. The continuum has not been subtracted from this p (see text).

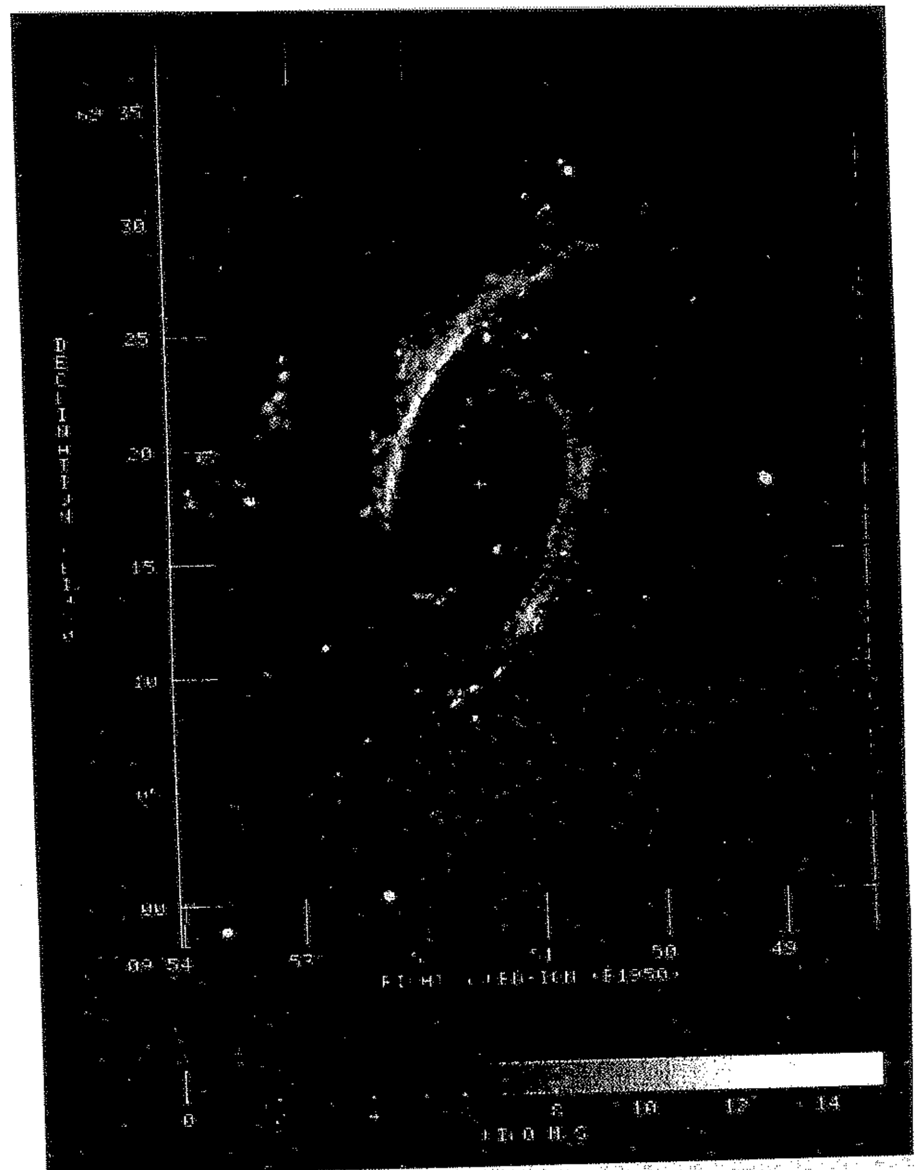


FIG. 6. Intensity-weighted velocity dispersion map. A cross marks the center of the galaxy.

M81 Adler and Westpfahl 1996

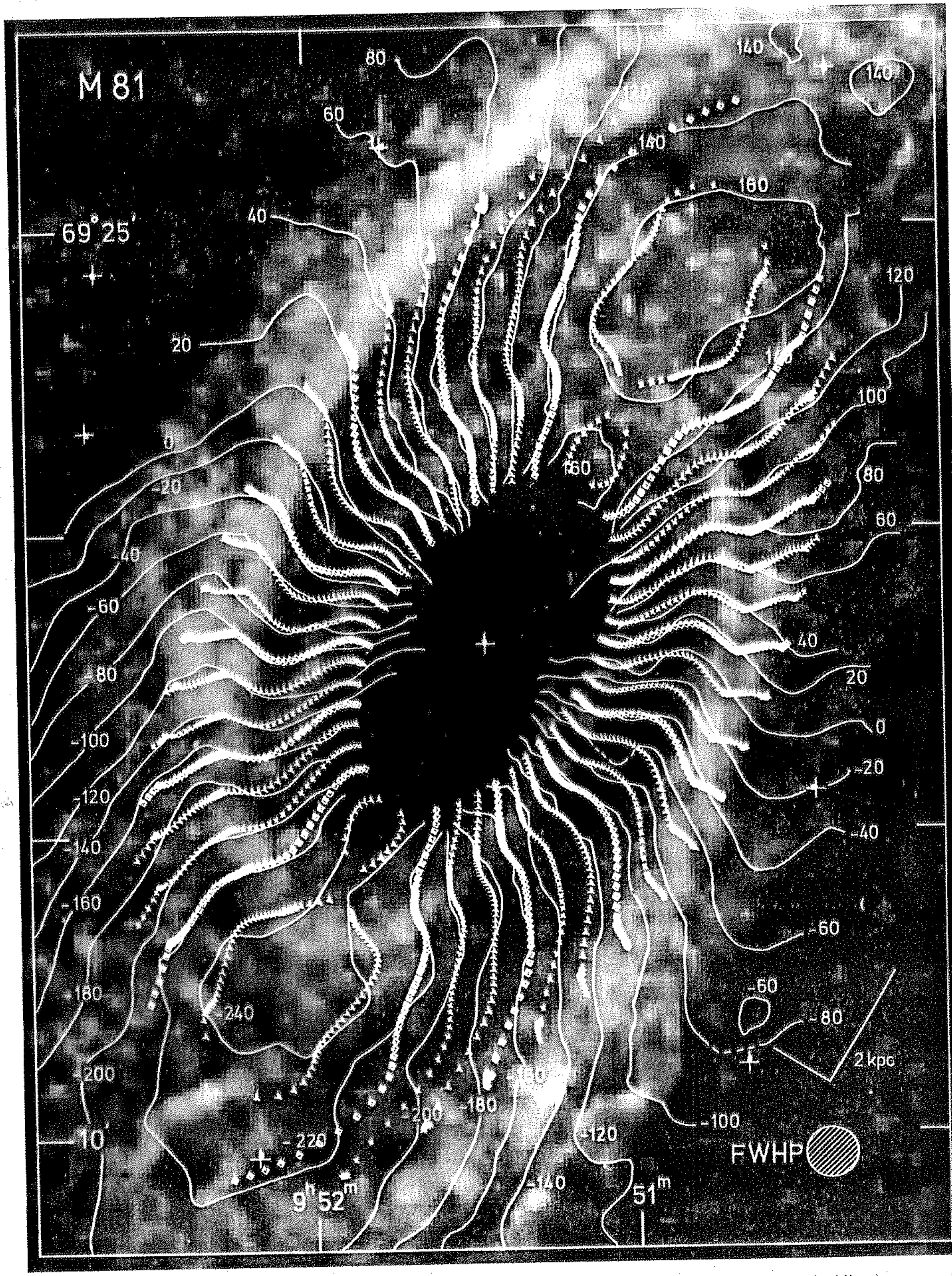


Fig. 5. The radial-velocity field of the final model (symbols) together with the observed velocity field (full and dashed lines) at an angular resolution of 50″, superimposed on a radiograph of the density distribution of the atomic hydrogen at 25″ resolution. See also the caption of Fig. 4

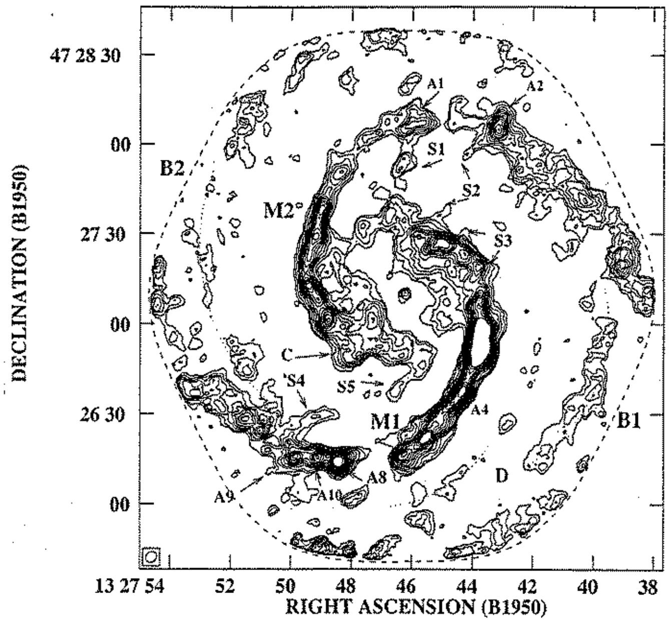


FIG. 1*a*

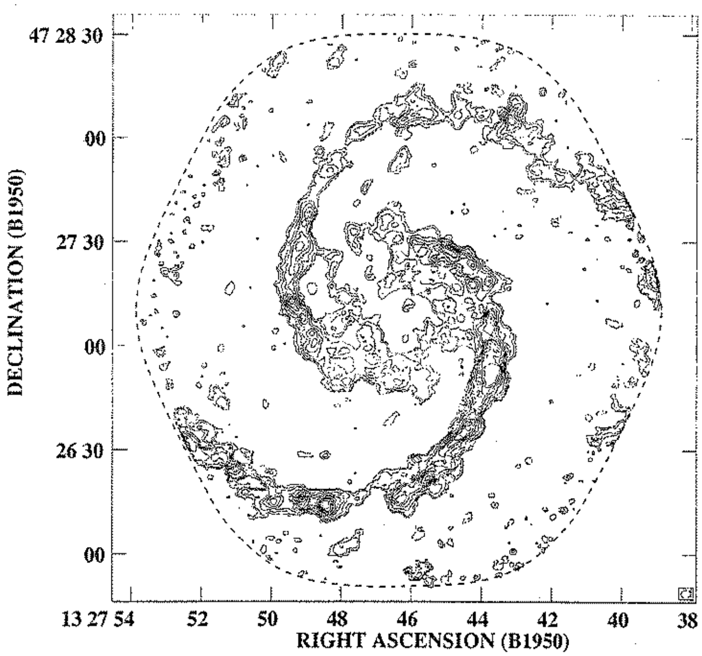


FIG. 1*b*

FIG. 1.—(*a*) CO 1–0 integrated intensity map, naturally weighted with synthesized beam 3".95 × 3".27 and beam P.A. = −42°.8. Contour levels are 1.1 Jy beam^{-1} km s^{-1} × (1.0, 2.5, 4.0, …, 17.5, 19.0). The peak flux is 29.04 Jy beam^{-1} km s^{-1} at $\alpha = 13^h27^m43^s.715$; $\delta = +47^{\circ}26'52''.25$ in the M1 arm. The lowest contour is at the 3 σ level. The total integrated flux in the map is 2.7 × 10^3 Jy km s^{-1}. The dotted lines mark the secondary arms B1 and B2, and the string of interarm D clouds (see text). Arrows indicate other features discussed in the text. The outer map cutoff is indicated with a dashed line. This cutoff is approximately 5" inside the map's outer primary beam half-power points. (*b*) CO 1–0 integrated intensity map robustly weighted with synthesized beam 2".88 × 2".11 and beam P.A. = −80°.7. Contour levels are 0.4 Jy beam^{-1} km s^{-1} × (1.0, 4.0, 8.0, …, 32). The peak flux is 18.64 Jy beam^{-1} km s^{-1} at the same position as for the naturally weighted map. The lowest contour is at the 3 σ level. The total integrated flux in the map is 2.0 × 10^3 Jy km s^{-1}. The outer map cutoff is indicated with a dashed line. (*c*) Position-velocity cut through the M1 and B1 features, at P.A. 242°. Zero velocity is at 472 km s^{-1}. (*d*) CO contours overlaid on an *HST* archive image of the center of M51 with Hα shown in red. The CO traces the main dust lanes, and the strong Hα is often seen on the downstream side of massive GMAs. The orientation angle of the image may be determined by comparison with *a*, which shows the CO contours on an equatorial coordinate reference frame.

M51 Aalto et al. 1999

ApJ 522, 165

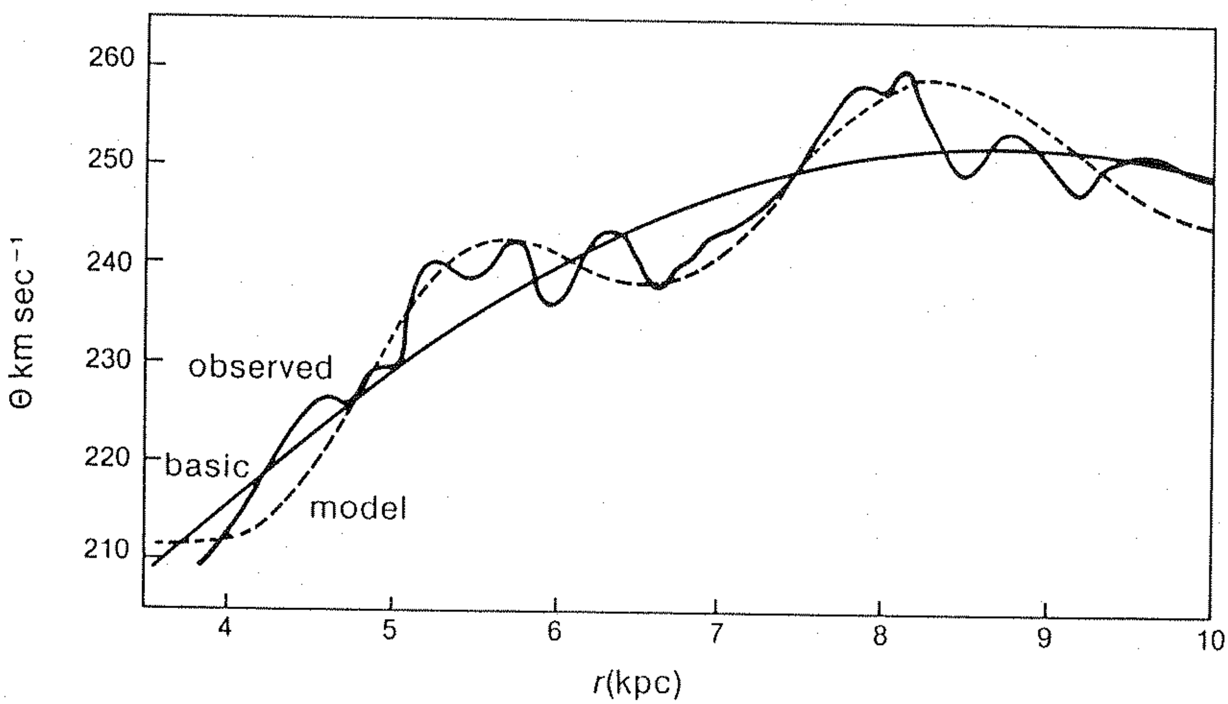


Figure 30.7. Rotation curve in disk with spiral arms. The basic (unperturbed) rotation curve is perturbed to give the dashed curve, as in equation (30.63). The observed rotation curve including spiral and local features is also shown.

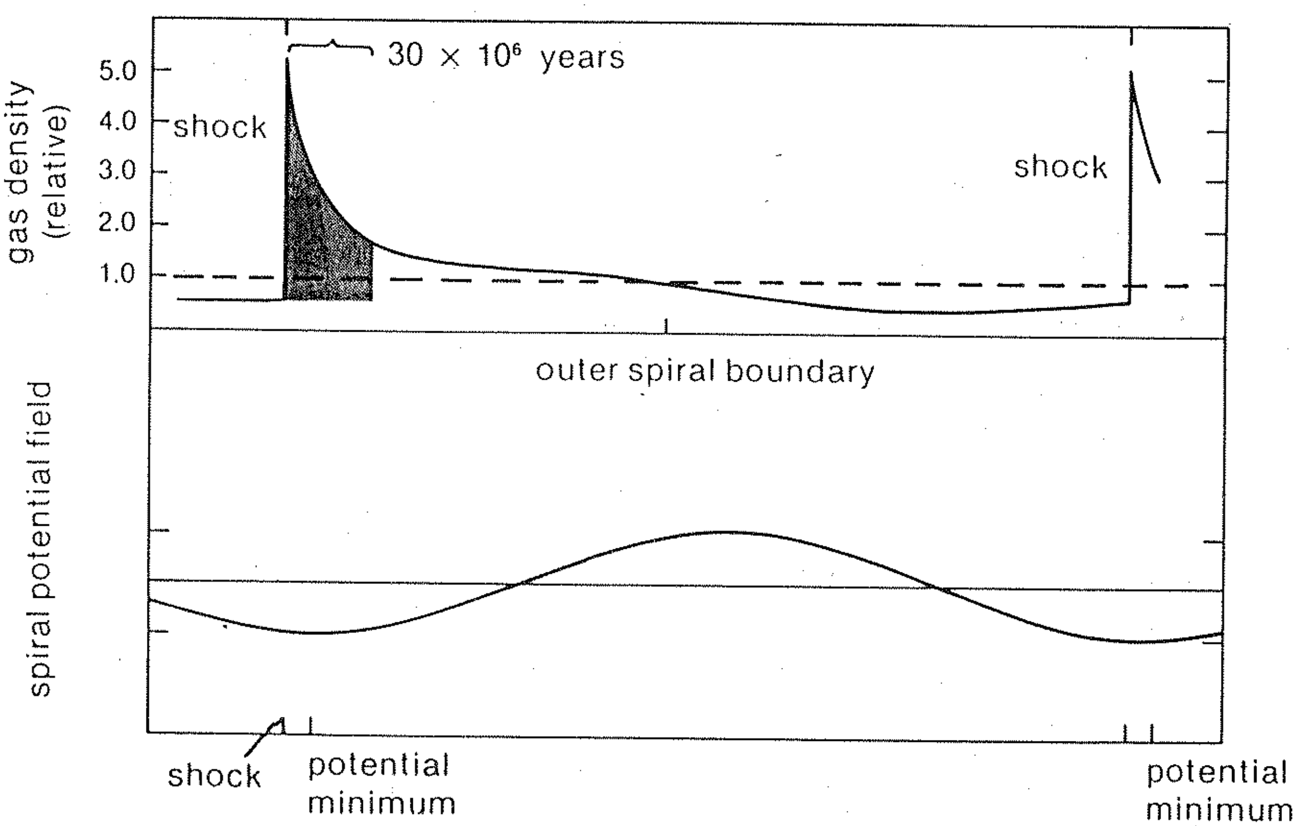


Figure 30.9. Gas density (upper) and spiral gravitational field (lower) in galactic disk versus distance normal to spiral arms. The shock front lies just inside the spiral potential minimum. Gas moves into the shock front (which may trigger star formation). Newly born stars and HII regions lie just behind the shock.

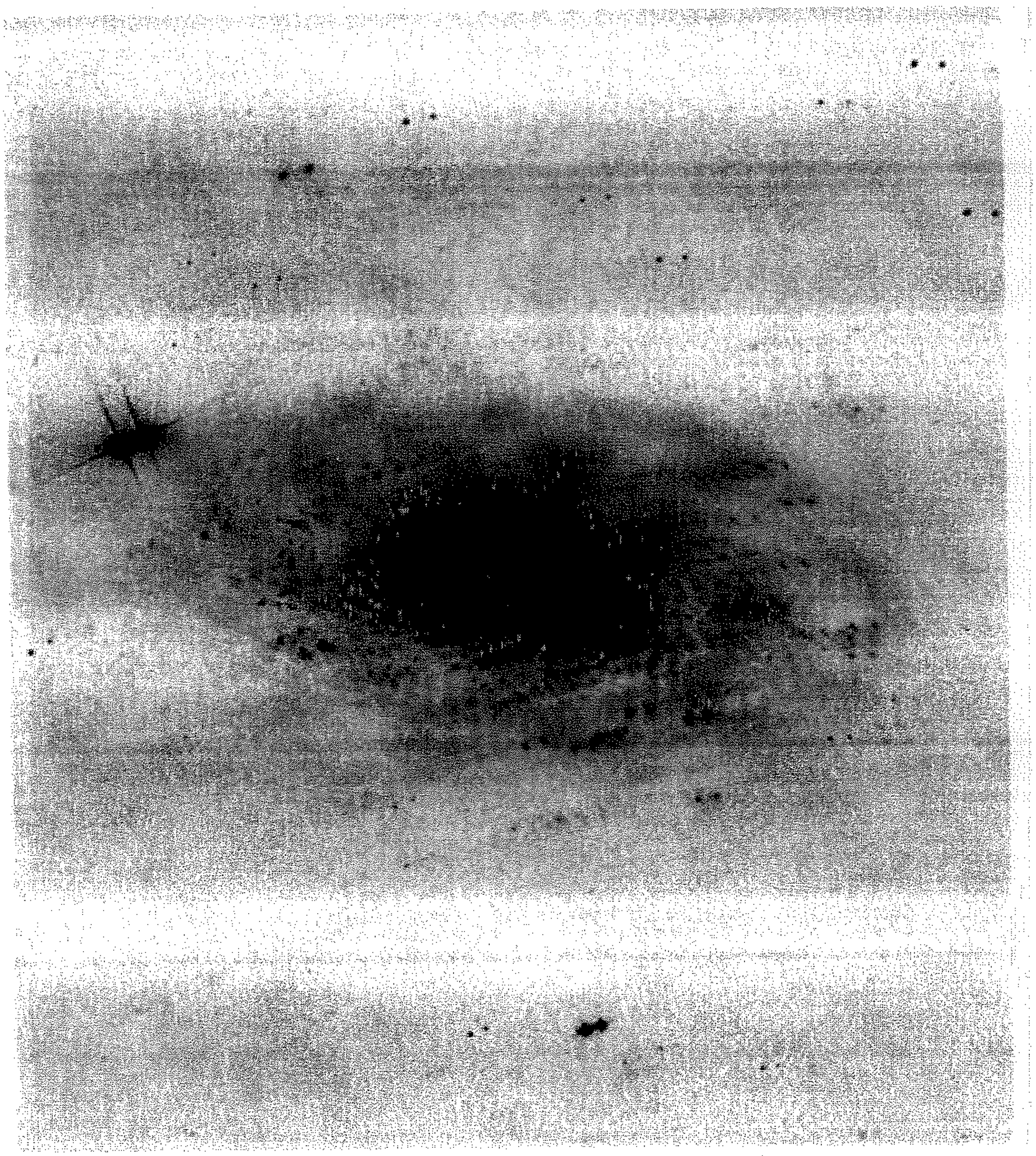
Stochastic, Self Propagating Star Formation

Some galaxies don't show organized "grand design" structure. Insted they show a multitude of short arm segments (e.g. NGC 5055).

If starformation can be modeled as a "chain reaction" phenomena, then differental rotation will shear this (SF) into short arm segments.

Gerola & Seiden 1978 ApJ **223**, 129

Seiden & Gerola 1979 ApJ **233**, 56



1979ApJ...233...56S

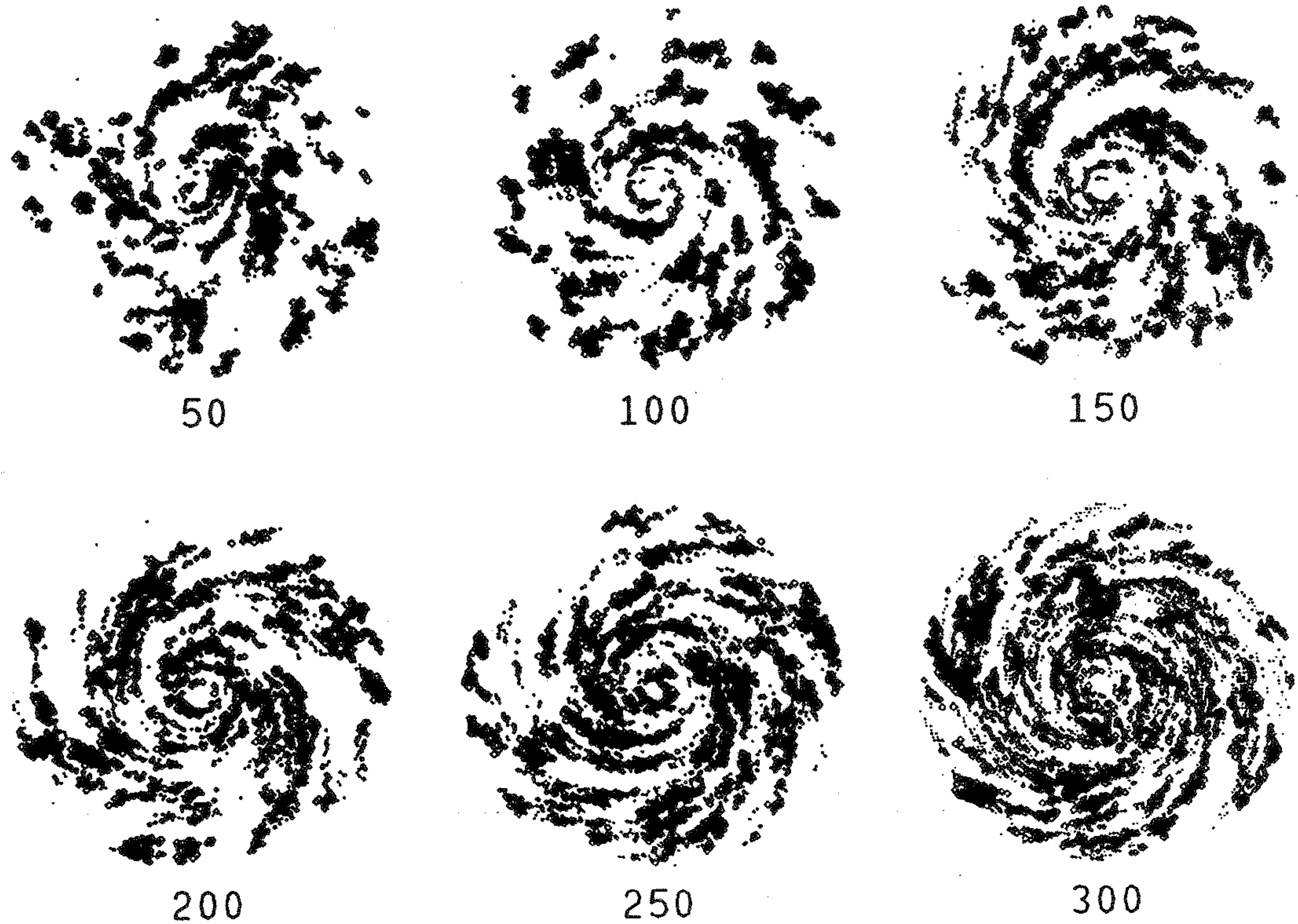


FIG. 1.—Model galaxies having flat rotation curves. The value of the velocity in km s^{-1} is given under each model.