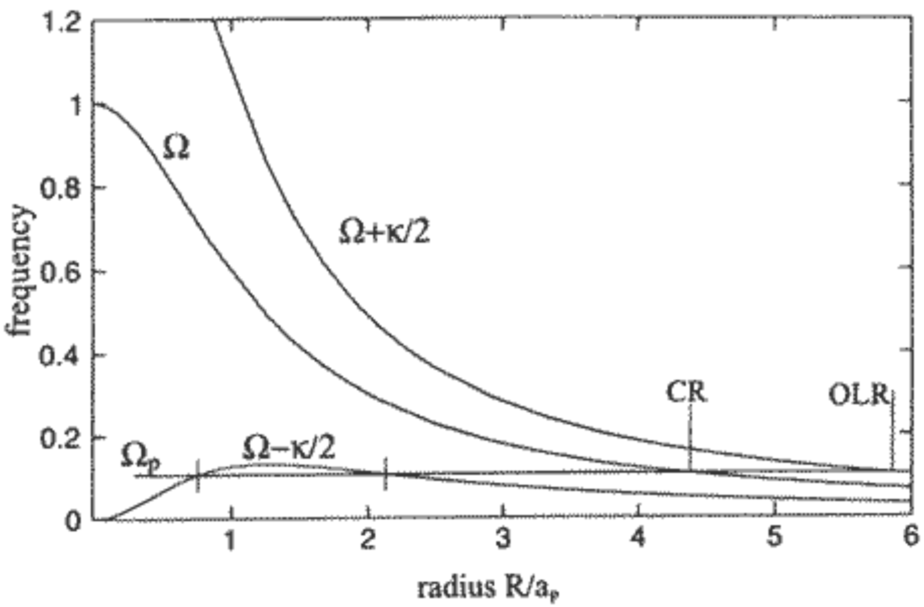
Astr 5465 April 3, 2020
Galactic Dynamics I: Disks Continued

- Galactic Disk Dynamics
  - Importance of Resonances
    - If stars undergo periodic perturbation at same phase in orbit
    - Similar to asteroids and Jupiter
    - Spiral pattern requires solid-body rotation and sprials have flat rotation curves
    - Solution to the "winding problem" is to have the spiral be a wave propagating through the disk
    - Star pass through the arms and vice versa
    - A pattern that rotates with $\Omega_p = \Omega - k/2$ will rotate as almost a solid body (Ω_p)
    - Spiral structure will exist for ILR < R < OLR
      ILR: $\Omega_p = \Omega^* - \kappa/2$, OLR: $\Omega p = \Omega^* + \kappa/2$
    - Outside that regions the stars are out of phase with the pattern and spiral density waves cannot exist
    - Within CR (co-rotation) stars lead pattern, outside CR the pattern leads the stars (see dust lanes in M51)



- Derivation of Density Wave Theory
  - Rather complicated and lengthy
    - Start with hydrodynamic equations and continuity equation
    - Assume a spiral form for the density perturbation
    - Show that the response of a star is an epicyclic orbit
  - I will write something up after spring break

Galactic Dynamics I: Disks Continued

- **Galactic Disk Stability**
  - **Absolutely cold disk will gravitationally collapse**
  - **Stellar velocity dispersion inhibits collapse**
  - **Differential rotation (shear) inhibits collapse**
  - **Toomre (1964) specifies conditions for disk stability via a Jeans argument:**

Consider an overdense region of radius R_J in a non-rotating disk:

The timescale for collapse:

$$t_{coll} \approx \frac{R_J}{\left(GM / R_J\right)^{1/2}} \approx \left(R_J^3 / GM\right)^{1/2}$$

Since the surface density $\Sigma \approx M / R_J^2$

$$t_{coll} \approx \left(R_J / G\Sigma\right)^{1/2}$$

Similar the timescale for a star to escape is:

$t_{esc} \approx R_J / \sigma$ where σ is the velocity dispersion.

Thus collapse will occur if $t_{coll} < t_{esc}$ or when:

$$\left(R_J / G\Sigma\right)^{1/2} < R_J / \sigma$$

The region will be stable if:

$$R_J < \sigma / G\Sigma$$

Now consider a rotating disk:

The local angular velocity is Oort's constant B and the region is stable if $F_{cent} > F_{grav}$ and so

$$RB^2 > GM / R^2 = G\Sigma$$

Thus the criteria for stability is:

$R_{rot} > G\Sigma / B^2$ or:

$R_J > R_{rot}$ or:

$\sigma^2 / G\Sigma > G\Sigma / B^2$ or $\sigma B / G\Sigma > 1$

but since $B = \kappa^2 / 4\Omega$ and $\kappa \approx 1\text{-}2\Omega$ then $B \approx \kappa / 3$

and our stability condition becomes

$$Q \equiv \frac{\sigma |B|}{G\Sigma} \cong \frac{\sigma\kappa}{3G\Sigma} > 1$$

Thus spiral structure can occur when:

Σ is high and σ is low or $|B|$ is low.

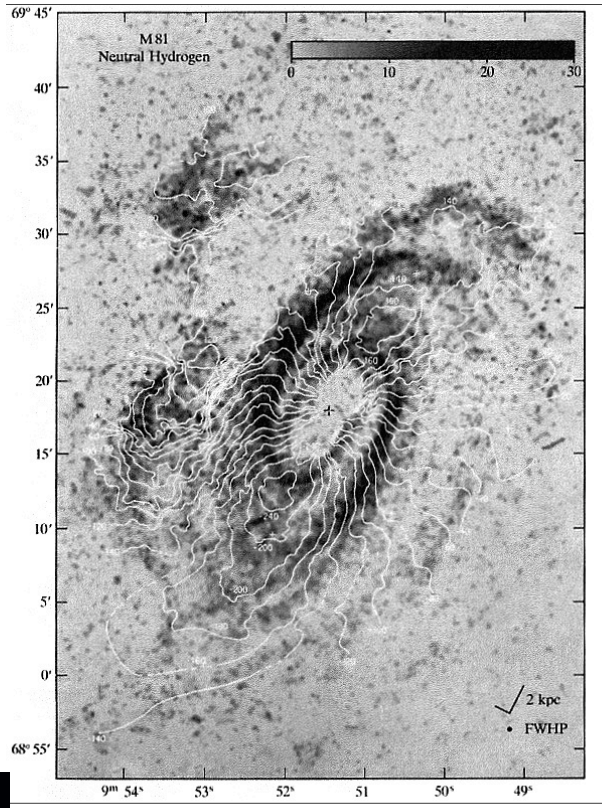
In the solar neighborhood $\sigma \approx$ 30 km/sec,

$\Sigma \approx 50\ \mathrm{M}_{sun}/\mathrm{pc}^2$, $\kappa \approx 36$ km/sec/kpc and so

Q ≈ 1.4 and locally stable

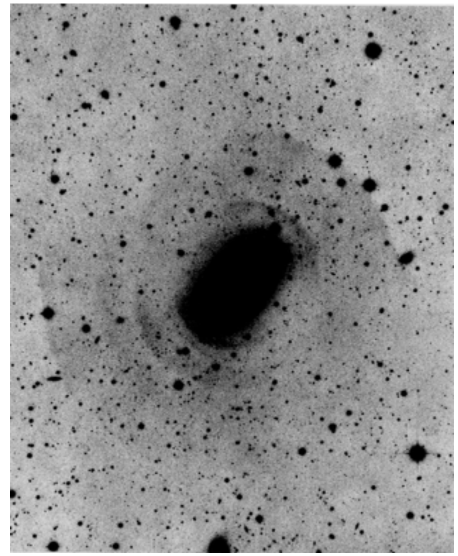
Galactic Dynamics I: Disks Continued

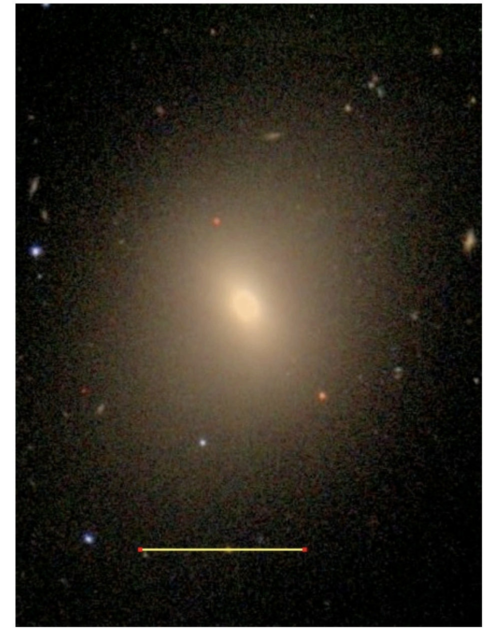
- **Comparison with Observations**
  - Density wave theory predicts perturbations near spiral arms
  - Velocity perturbations of ~ 20 km/sec
  - 2-d HI maps ideally sample gas response
  - M81 maps with VLA (Westfall et al.) show clear signature
  - Some galaxies don’t show classic two-armed spirals (flocculent)
  - Some other mechanism at work?
  - Spiral features could be short-lived due to rotational shear (segments come and go)
  - Self-propagating star formation?



Galactic Dynamics II: Ellipticals

- **The 3-d nature of elliptical galaxies complicates their dynamics**
  - **Intrinsic shape originally thought to result from rotation**
  - **Oblate: frisbee-like, Prolate: football-like**
  - **Intrinsic shapes like neither but triaxial (oblate-like)**
  - **Now we know that the shapes arise from non-isotropic velocity distribution functions**
  - **Not surprising given their origin via mergers**
  - **Direct evidence from shells and complex structure**
  - **Some ellipticals don't even have elliptical isophotes:**
  - **Boxy: squareish isophotes, Disky: pointy isophotes**
  - **Twisting isophotes direct signature of triaxial shape**

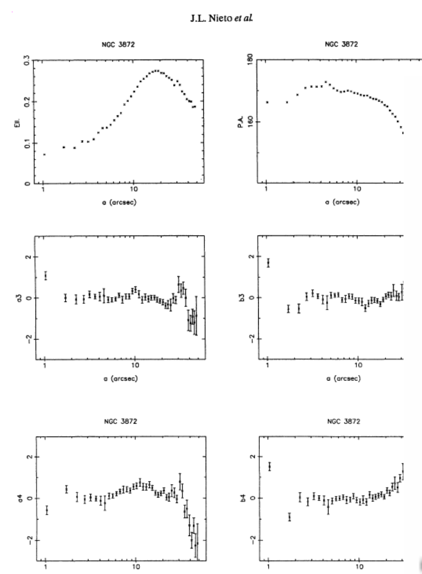




We can parameterize isophote shape as a Fourier expansion:

$$R(\phi) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\phi) + \sum_{n=1}^{\infty} b_n \sin(n\phi)$$

a_0 is the mean radius, a_1, b_1 define center, a_2, b_2 define ellipt. and pos. angle, a_3 b_3 measure asymmetries (dust), a_4 defines boxyness or diskyness: $a_4 > 0$: disky, $a_4 < 0$: boxy.

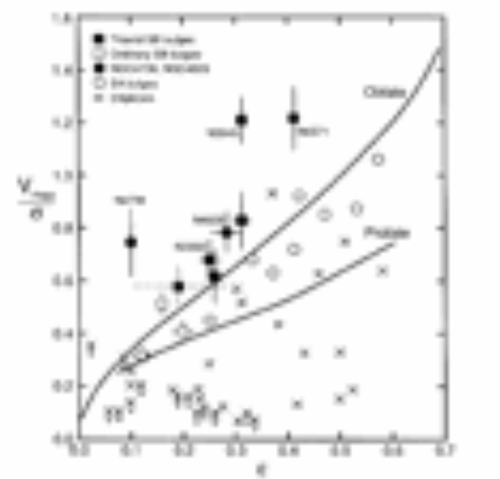


- **Rotation flattening predicts:**

$$\left(\frac{V_r}{\sigma}\right) = \left[\frac{1-b/a}{b/a}\right]^{1/2} = \left[\frac{\varepsilon}{1-\varepsilon}\right]^{1/2}$$

To compare with observations define:

$$\left(\frac{V_r}{\sigma}\right)^* = \left(\frac{V_r}{\sigma}\right)_{obs} / \left(\frac{V_r}{\sigma}\right)_{expect} = \left(\frac{V_r}{\sigma}\right)_{obs} / \left[\frac{\varepsilon}{1-\varepsilon}\right]^{1/2}$$



Results suggest rotational support for low lum. Es but inconsistent with rotational support for luminous Es.

Galactic Dynamics II: Analytic Models for Ellipticals

- **Potential Theory**
  - **Scalar nature of potential means we can add components.**
  - **Relate density to potential via Poisson's equation**
  - **Density – potential pairs of analytic functions**
  - **We can go back and forth via Poisson's equation**

  (see Binney & Tremaine Ch. 4 for details)

$$\Phi(r) = -G\int_V \frac{\rho(r')}{|r'-r|}d^3r' \quad \text{and} \quad F(r) = -\nabla\Phi(r) \quad \text{with}$$

$$\nabla\cdot F(r) = -4\pi G\rho(r) \quad \text{and} \quad \nabla^2\Phi(r) = 4\pi G\rho(r)$$

Some example density-potential pairs:

Point Mass (Keplerian Potential):

$$\Phi(r) = -GM/r \quad \text{so} \quad F(r) = -\nabla\Phi = -GM/r^2$$

$$V_c^2(r) = GM/r \quad \text{and} \quad V_{esc}^2(r) = 2GM/r$$

Finite Homogenous Sphere

(radius = a, ρ(r) = const for r < a):

$$r < a : \Phi(r) = -2\pi\rho(a^2 - r^2/3) \text{ so } F(r) = -GM(r)/r^2$$

$$r > a : \Phi(r) = GM/r \text{ (Keplerian)}$$

Simple Harmonic Motion with $P_r = (3\pi/G\rho)^{1/2}$

$$V_c(r) = [(4/3)\pi G\rho]^{1/2} r \quad \text{(solid body rotation)}$$

Logarithmic Potential (Flat Rotation Curve):

$$F(r) = \frac{V_c^2}{r} = -\frac{d\Phi}{dr} \quad \text{so} \quad \Phi(r) = V_c^2 \ln r + const.$$

Power Law Spherical Systems:

$$\rho(r) = \rho_0 (r/a)^{-\alpha} \quad \text{so} \quad M(<r) = \frac{(4\pi G a^3 \rho_0)}{(3-\alpha)}(r/a)^{3-a}$$

$$\Phi(r) = -\frac{(4\pi G a^2 \rho_0)}{(3-\alpha)(\alpha-2)}(r/a)^{2-a} = V_c^2/(\alpha-2)$$

Note: for $\alpha > 3$: M(< r) $\rightarrow \infty$ (infinite mass at center)

for $\alpha < 3$: M(∞) $\rightarrow \infty$ (total mass diverges)

Special case of $\alpha = 2$: singular isothermal sphere:

$$\Phi(r) = 4\pi G a^2 \rho_0 \ln(r/a) \quad \text{and} \quad V_c = (4\pi G \rho_0 a^2)^{1/2} = const.$$

More complex models include those of Hernquist and Jaffe:

$$\rho_H(r) = \left(\frac{Ma}{2\pi r(r+a)^3}\right) \quad \text{with} \quad \Phi_H(r) = -\frac{GM}{(r+a)}$$

$$\rho_J(r) = \left(\frac{Ma}{4\pi r^2(r+a)^2}\right) \quad \text{with} \quad \Phi_J(r) = \frac{GM}{a}\ln\left(\frac{r}{r+a}\right)$$

Plummer Sphere is analytic solution for hydrostatic equilibrium:

$$\rho_P(r) = \left(\frac{3M}{4\pi b^3}\right)\left(1+\frac{r^2}{b^2}\right)^{-5/2} \quad \text{with } \Phi_P(r) = -\frac{GM}{\sqrt{(r^2+b^2)}}$$

Galactic Dynamics III: CBE Models for Ellipticals

- **Derivation of the Collisionless Boltzman Equation**
  - **Consider the crossing times:** $t \sim \frac{R}{V}$**, For a uniform sphere:**
  - $\mathbf{V} = \sqrt{\frac{GM}{R}}, \rho = \frac{3M}{4\pi R^3}, \ \mathbf{t} = \sqrt{\frac{3}{4\pi G\rho}}, \sim 10^8$ **years**
  - **For two stars to encounter each other at sufficiently small impact parameter to significantly change their velocities is the two body relaxation time:**

$t_{relax} \sim \left(\frac{R^3}{GM}\right)^{1/2} \frac{N}{8 \ln N} \sim 10^9$ **years for globular clusters and** $\sim 10^{12}$ **years for galaxies, so what is the mechanism?**

First consider "collisionless" sysstems, i.e., no star-star interactions, only smooth background potential. The distribution function (DF) describes the phase space density: $f(r,v,t)d^3r d^3v$ = number of stars at $\vec{r}$ with $\vec{v}$ at time t in the range d^3r and d^3v. If we treat the system as a fluid we can make use of the continuity equation: The net flow in the x coordinate over interval dt is:

$$v_x dt dv_x [f(x, v_x, t) - f(x + dx, v_x, t)] = -v_x dt dv_x \frac{\partial f}{\partial x} dx$$

The net flow from the velocity gradient is:

$$dx \frac{dv_x}{dt} dt [f(x, v_x, t) - f(x, v_x \_ dv_x, t)] = -dx dt \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} dv_x$$

Summing these is the net change within the region:

$$dx dv_x \frac{\partial f}{\partial t} dt = -dt dx v_x \frac{\partial f}{\partial x} dv_x - dx dt \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} dv_x$$

dividing by $dx dv_x dt$ gives:

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + \frac{dv_x}{dt} \frac{\partial f}{\partial v_x} = 0$$

but since:

$$\frac{dv_x}{dt} = a_x = -\frac{\partial \Phi}{\partial x} \text{ we have:}$$

$$\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} - \frac{\partial \Phi}{\partial x} \frac{\partial f}{\partial v_x} = 0$$

Adding in the y and z dimensions gives:

$$\frac{\partial f}{\partial t} + \vec{v} \cdot \nabla f - \nabla \Phi \cdot \frac{\partial f}{\partial \vec{v}} = 0$$

This is the collisionless Boltzman Equation.
Note that the phase space density is constant. As real space density increases so does σ and vice versa. By itself the CBE is not very useful. We need a way to relate a model to observed quantities like density and velocity dispersion. We do this by taking moments of the CBE.

Galactic Dynamics III: Models for Ellipticals

For the one dimensional case if take the 0-th moment by integrating the CBE over all v_x :

$$\frac{\partial n}{\partial t}+\frac{\partial\left(n\langle v_x\rangle\right)}{\partial x}=0$$ with $n\equiv n(x,t)$ being the space density of stars and $\langle v_x\rangle$ being the average velocity along x. If we multiply the CBE by v_x and integrate over all v_x (1-st moment) and using the equation above we get:

$$\frac{\partial\langle v_x\rangle}{\partial t}+\langle v_x\rangle\frac{\partial\langle v_x\rangle}{\partial x}=-\frac{\partial\Phi}{\partial x}-\frac{1}{n}\frac{\partial\left(n\sigma_x^2\right)}{\partial x}$$

In three dimensions this becomes the Jeans Equation:

$$\frac{\partial\langle v_j\rangle}{\partial t}+\langle v_i\rangle\frac{\partial\langle v_j\rangle}{\partial x_i}=-\frac{\partial\Phi}{\partial x_j}-\frac{1}{n}\frac{\partial\left(n\sigma_{i,j}^2\right)}{\partial x_i}$$

This is similar to Newton's 2-nd law but with $n\sigma_{i,j}^2$ as a stress tensor analogous to an anisotropic pressure. For a spherical, steady-state system (1-st term 0) we have:

$$\frac{1}{n}\frac{d\left(n\sigma_r^2\right)}{dr}+\frac{1}{r}\left[2\sigma_r^2-(\sigma_\theta^2+\sigma_\phi^2\right]-\frac{\langle v_\phi\rangle^2}{r}=-\frac{d\Phi}{dr}$$

Introducing anisotropy parameters: $\beta_\theta=1-\sigma_\theta^2/\sigma_r^2$ and $\beta_\phi=1-\sigma_\phi^2/\sigma_r^2$ and using $2\beta=\beta_\theta+\beta_\phi$ and $V_{rot}=\langle v_\phi\rangle$

$$\frac{1}{n}\frac{d\left(n\sigma_r^2\right)}{dr}+2\beta\frac{\sigma_r^2}{r}-\frac{V_{rot}^2}{r}=-\frac{d\Phi}{dr}$$

- **Models Built from Distribution Functions**
  - **Alternative to Analytic Model for Potential**
  - **One of the best use of the CBE is to build a model from a distribution function.**
  - **We'll assume the distribution function is directly tied to the density distribution, i.e., stars are not just a tracer population.**

Integrating the distribution function yields the potential: $4\pi\int f(\vec{r},\vec{v})d^3\vec{v}=4\pi G\rho(\vec{r})=\nabla^2\Phi(\vec{r})$ assuming stars have a particular mass. Adopting a DF in energy (E) and angular momentum ($|L|$) and the spherical form of ∇^2 :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right)=4\pi G\int f\left(\frac{v^2}{2}+\Phi,\left|\vec{r}\times\vec{v}\right|\right)d^3\vec{v}$$

We now at least have a self-consistent model. For a spherical, isotropic system the DF = f(E) only so:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Psi}{dr}\right)=-16\pi^2G\int_0^{\sqrt{2\Psi}}f(\Psi-v^2/2)v^2dv$$

where $\Psi=\Phi_0-\Phi$ is the relative potential. This equation is the starting point for building a "simple" model. Example1: Polytropic Sphere with power law f(E_r) :

Galactic Dynamics III: Models for Spirals

- **Potential Theory for disks**
  - **Not relevant to ellipticals but this is aa good a place for it as any.**
  - **Disks are generally more complicated since lack of spherical symmetry means mass at r > r_0 contributes**
  - **As before we might search for analytic density-potential pairs (see right-hand panel)**
- **Poisson's Equation (Gauss' Law for Mass):**
  $$\nabla^2\phi = 4\pi G\varrho$$
- **Another approach is to use the distribution of stars in phase space to fully describe the system. This approach doesn't require an assumption of a steady state. We can include two-body relaxation and other dissipative phenomena to evolve the system over time. The downside is that it is a bit mathematical. But with cylindrical symmetry the**
- **Collisionless Boltzman Equation Becomes:**
  $$\partial^2\phi$$

Mestel's Disk:

$\Sigma(r) = \Sigma_0(r/r_0)$ with $V_c^2(r) = 2\pi G\Sigma_0 r_0 = const.$

Exponential Disk:

$\Sigma(r) = \Sigma_0 e^{-r/r_d}$ where r_d is the disk scale length with:

$$V_c^2(r) = 4\pi G\Sigma_0 r_d y^2\left[I_0(y)K_0(y) - I_1(y)K_1(y)\right]$$

where $y = \dfrac{r}{2r_d}$ and I_n and K_n are Bessel functions of the 1-st and 2-nd kind. Another disk pair is that of Kuzmin:

$$\Sigma_K(r) = \frac{aM}{2\pi(r^2+a^2)^{3/2}} \quad \text{with} \quad \Phi_K(r,z) = -\frac{GM}{\sqrt{r^2+(a+|z|)^2}}$$

Toomre disks are a series derived by differentiating Kuzmin disks (n=1: Kuzmin disk, n=∞: Gaussian disk):

$$\Sigma_{T_n}(r) = \left(\frac{d}{da}\right)^{n-1}\Sigma_K(r) \quad \text{with} \quad \Phi_{T_n}(r,z) = \left(\frac{d}{da}\right)^{n-1}\Phi_K(r,z)$$

For a spiral galaxy we could add the potentials of a bulge and a disk but we might also consider flattened potenitals. Miyamoto-Nagai proposed:

$$\rho_M(r,z) = \left(\frac{Mb^2}{4\pi}\right)\frac{ar^2+(a+3B)(a+B)^2}{[r^2+(a+B)^2]^{5/2}B^3} \quad \text{with}$$

$$\Phi_M(r,z) = -\frac{GM}{\sqrt{r^2+(a+B)^2}} \quad \text{and} \quad B^2 = z^2+b^2$$

Note that these reduce to a plummer model if a = 0 and to the Kuzmin disk if b = 0

Galactic Dynamics III: Models for Spirals

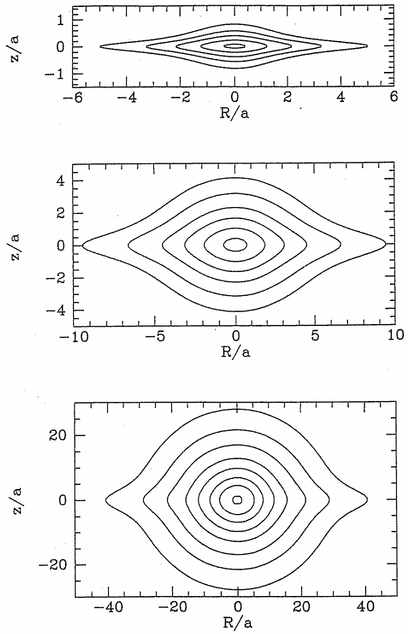


Figure 2-6. Contours of equal density in the (R, z) plane for the Miyamoto-Nagai density distribution (2-50b) when: $b/a = 0.2$ (top); $b/a = 1$ (middle); $b/a = 5$ (bottom). Contour levels are: $(0.3, 0.1, 0.03, 0.01, \ldots)M/a^3$ (top); $(0.03, \ldots)M/a^3$ (middle); $(0.001, \ldots)M/a^3$ (bottom).