Astr 5465 Mar. 29, 2018
Galactic Dynamics I: Disks

- **Two Body Problem Review**
- This lecture summarizes the two body problem from classical mechanics but from the perspective of
- potential theory. Good references are Thornton & Marion (ch. 8) and Goldstein, Poole and Safko (ch. 3).
- We assume that two point masses are moving under the influence of a mutual central force.
- **The Equivalent One-body Problem – the Reduced Mass**
- We begin by considering a system of two point masses, m_1 and m_2 with the potential energy (U) being only a function of the separation between the two masses: $r = |_{r_1} - r_2|$. In terms of the Lagrangian:
- $L = \frac{1}{2} m_1 |\dot{r}_1|^2 + \frac{1}{2} m_2 |\dot{r}_2|^{2^{+U(r)}}$
- Placing the origin at the center mass allows us to consider the relative motion ($r = r_1 + r_2$):
- $m_1 r_1 + m_2 r_2 = 0$ such that $r_1 = \frac{m_2}{m_1+m_2}$r and $r_2 = \frac{m_1}{m_1+m_2}$r. The Lagrangian then becomes:
- $L = \frac{1}{2}\mu|\dot{r}|^2 - U(r)$ where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

- **Conservation Theorems are known as Integrals of Motion**
- The first is the conservation of momentum:
- $\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = const.$ If we write the Lagrangian in polar coordinates we have:

- $L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) \quad - U(r)$ but the angular momentum in the θ coordinate is conserved:

- $\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}}$ and so $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2 \dot{\theta} = const.$

- The quantity p_θ is known as the first integral of motion and is usually denoted as l and thus:
- $\boxed{l = \mu r^2 \dot{\theta} = const.}$ This is interpreted in terms of the area swept out over an interval dt:

- $A = \frac{1}{2}r^2\theta$ and so $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2\mu} = const.$ so that the areal velocity is constant in time (Kepler's Second Law of Planetary motion). Another first integral of motion comes from the conservation of the total energy:

- $T = U = E = const.$ with $E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r) \quad or\!:\; E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r).$
- All that is needed is the form of *U(r)* and integration yields r as a function of *E* and *l*.

- **The Equations of Motion**
- Solving the above equation for $\dot{r}$ yields:
- $\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu r^2}}$
- Since we can write $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$ we then substitute $\dot{\theta} = {}^{l}/_{\mu r^2}$ and the above for $\dot{r}$ we have:
- $\theta(r) = \int \frac{\left({}^{l}/_{r^2}\right)dr}{\sqrt{2\mu\left(E-U-\frac{l^2}{2\mu r^2}\right)}}$
- And together these are the equations of motion. However, this is not a particularly useful form, $\theta(t)$ is preferred as we will see below. Note that above we combined conservation of energy with conservation of momentum but we need the form of the potential in order to solve the problem. Put another way, we need to know the form of the force law $F(r) \propto r^n$ and analytical solutions are present only for n = 1, -2, -3. Another approach to deriving the equations of motion makes use of Lagrange's equation (see Marion and Thorton) and this is useful for determining $F(r)$ given an orbit, $r(\theta)$. One interesting result of this equation is the question of when orbits are closed. Since:
- $\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{\left({}^{l}/_{r^2}\right)dr}{\sqrt{2\mu\left(E-U-\frac{l^2}{2\mu r^2}\right)}}$
- In order for the orbit (path) to be closed $\Delta\theta$ needs to be a multiple of 2π. Specifically:
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- $\Delta\theta = 2\pi\frac{a}{b}$ where a and b are integers (the orbit can be like a Lissajous figure). This concept is
- central to the idea of orbital resonance.

- **Orbits in a Central Field**
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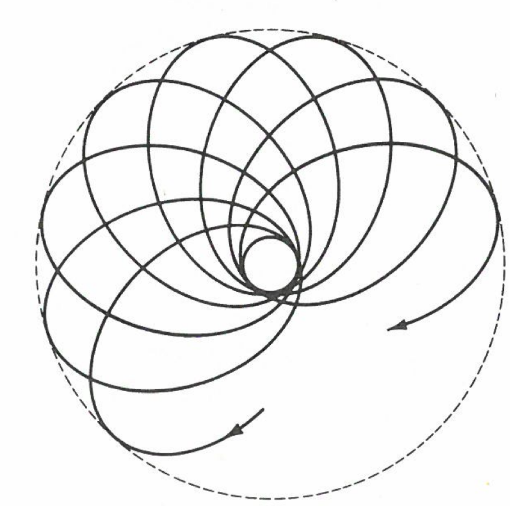
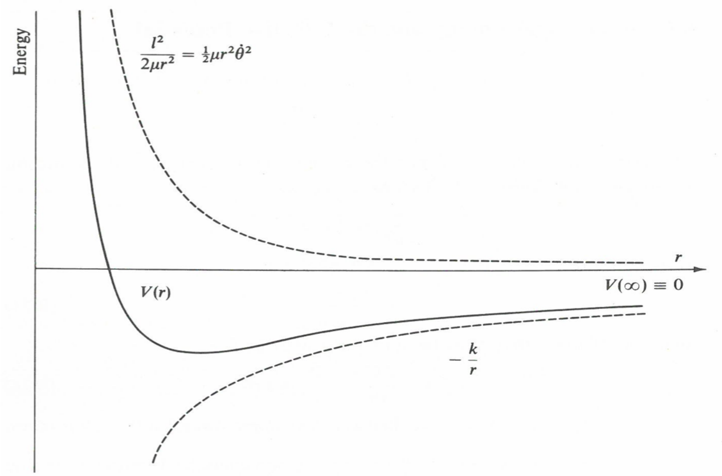
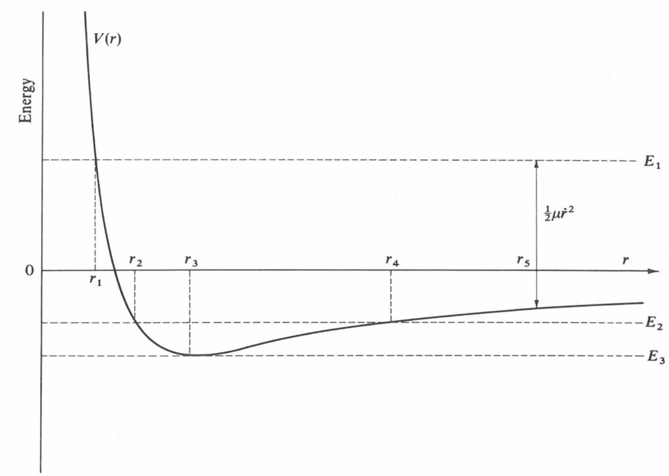
- We now continue to follow the traditional approach and consider orbits in a central field. Recall:
- $\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu r^2}}$
- Thus $\dot{r}$ will be zero at the "turnaround radii", r_{min} and $r_{max.}$ This will occur when:
- $E - U(r) - \frac{l^2}{2\mu r^2} = 0$. Note that for certain values of *E, U(r)* and *l* only a single value of *r* is allowed (single root) and the orbit is then circular.
- **Centrifugal Energy and the Effective Potential**
- Recall that:
- $\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}$
- If we interpret this as a "potential energy" or an effective potential:
- $U_c = \frac{l^2}{2\mu r^2}$ then the corresponding force would be $F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^2} = \mu r\dot{\theta}^2$
- or the centrifugal force. We next introduce the idea of an effective potential:
- $V(r) = U(r) + \frac{l^2}{2\mu r^2}$
- For the gravitational potential $U(r) = -\frac{k}{r}$ we then have:
- $V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$
- Thus, we see that if the total energy is negative the particle is bound within the
- "turning points" and if the total energy is positive it is unbound.




Energy
$\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}^2$
r
$V(r)$
$V(\infty) \equiv 0$
$-\frac{k}{r}$


$V(r)$
Energy
E_1
$\frac{1}{2}\mu\dot{r}^2$
0
r_1
r_2
r_3
r_4
r_5
r
E_2
E_3