Astr 5465 Mar. 11, 2020

Galactic Dynamics I: Disks

- **Galactic Dynamics**
  - **Two Body Problem Overview**
  - **Subject is complex but we will hit the highlights**
    - **Our goal is to develop an appreciation of the subject which we can use to interpret observational data**
    - **See Binney & Tremaine and references therein for a full treatment of the subject**
  - **Distribution of mass determines gravitational acceleration on each star**
    - **Acceleration at each 3d point and 3d velocity constitutes 6 parameters in phase space which define the motion of each star at one point in time**
    - **We need to integrate these over time to determine stellar trajectory (orbit)**
    - **Following 10^{11} stars via an N-body simulation is currently impossible**
  - **Instead let's consider analytic models**
    - **Distribution of stars reflects their orbital trajectories over a long time interval.**
    - **So we assume a steady state for now**
- **Disk galaxies**
  - **Key concepts include**
    - **Circular motion**
    - **Deviations from circular motion**
    - **Resonances**
    - **Density Waves**
    - **Instabilities**

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Galactic Dynamics I: Disks

This lecture summarizes the two body problem from classical mechanics but from the perspective of potential theory. Good references are Thornton & Marion (ch. 8) and Goldstein, Poole and Safko (ch. 3). We assume that two point masses are moving under the influence of a mutual central force.

- **The Equivalent One-body Problem – the Reduced Mass**

Consider a system of two point masses, m_1 and m_2 with the potential energy (U) being only a function of the separation between the two masses: $r = |r_1\text{-}r_2|$. In terms of the Lagrangian:

$L = \frac{1}{2}m_1|\dot{r_1}|^2 + \frac{1}{2}m_2|\dot{r_2}|^2 + \mathrm{U(r)}$

Placing the origin at the center mass allows us to consider the relative motion ($r = r_1 + r_2$):

$m_1 r_1 + m_2 r_2 = 0$ such that $r_1 = \frac{m_2}{m_1+m_2}$ and $r_2 = \frac{m_1}{m_1+m_2}$r. The Lagrangian then becomes: $L = \frac{1}{2}\mu|\dot{r}|^2 - U(r)$ where $\mu = \frac{m_1 m_2}{m_1+m_2}$ is the reduced mass.

- **Conservation Theorems are known as Integrals of Motion**

The first is the conservation of momentum:

$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = const.$ If we write the Lagrangian in polar coordinates we have:

$L = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right)^2 - U(r)$ but the angular momentum in the θ coordinate is conserved:

$$\dot{p}_\theta = \frac{\partial L}{\partial \theta} = 0 = \frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} \text{ and so } p_\theta = \frac{\partial L}{\partial \dot{\theta}} = \mu r^2\dot{\theta} = const.$$

The quantity p_θ is known as the first integral of motion and is usually denoted as l and thus: $l = \mu r^2\dot{\theta} = const.$ This is interpreted in terms of the area swept out over an interval dt: $\mathrm{A} = \frac{1}{2}r^2\theta$ and so $\frac{dA}{dt} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2\mu} = const.$ so that the areal velocity is constant in time (Kepler's Second Law of Planetary motion). Another first integral of motion comes from the conservation of the total energy:

$T = U = E = const.$ with $E = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + U(r)$ or: $E = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} + U(r)$. All that is needed is the form of $U(r)$ and integration yields r as a function of E and l.

- **The Equations of Motion**
- Solving the above equation for $\dot{r}$ yields:
- $\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E - U) - \frac{l^2}{\mu r^2}}$
- Since we can write $d\theta = \frac{d\theta}{dt}\frac{dt}{dr}dr = \frac{\dot{\theta}}{\dot{r}}dr$ we then substitute $\dot{\theta} = {}^{l}/_{\mu r^2}$ and the above for $\dot{r}$ we have:
- $\theta(r) = \int \frac{\left({}^{l}/_{r^2}\right)dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$
- And together these are the equations of motion. However, this is not a particularly useful form, $\theta(t)$ is preferred as we will see below. Note that above we combined conservation of energy with conservation of momentum but we need the form of the potential in order to solve the problem. Put another way, we need to know the form of the force law $F(r) \propto r^n$ and analytical solutions are present only for n = 1, -2, -3. Another approach to deriving the equations of motion makes use of Lagrange's equation (see Marion and Thorton) and this is useful for determining $F(r)$ given an orbit, $r(\theta)$. One interesting result of this equation is the question of when orbits are closed. Since:
- $\Delta\theta = 2\int_{r_{min}}^{r_{max}} \frac{\left({}^{l}/_{r^2}\right)dr}{\sqrt{2\mu\left(E - U - \frac{l^2}{2\mu r^2}\right)}}$
- In order for the orbit (path) to be closed $\Delta\theta$ needs to be a multiple of 2π. Specifically:
-
- $\Delta\theta = 2\pi\frac{a}{b}$ where a and b are integers (the orbit can be like a Lissajous figure). This concept is
- central to the idea of orbital resonance.

- **Orbits in a Central Field**
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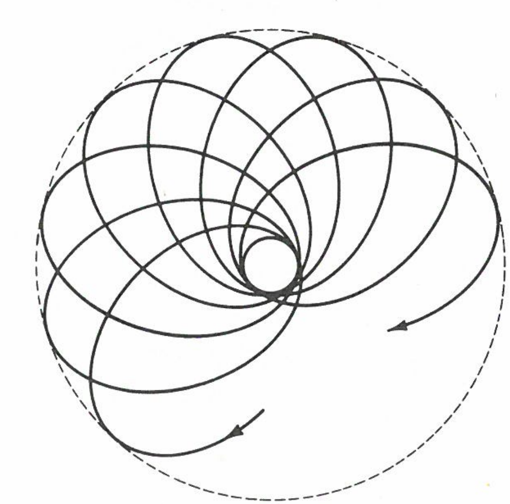
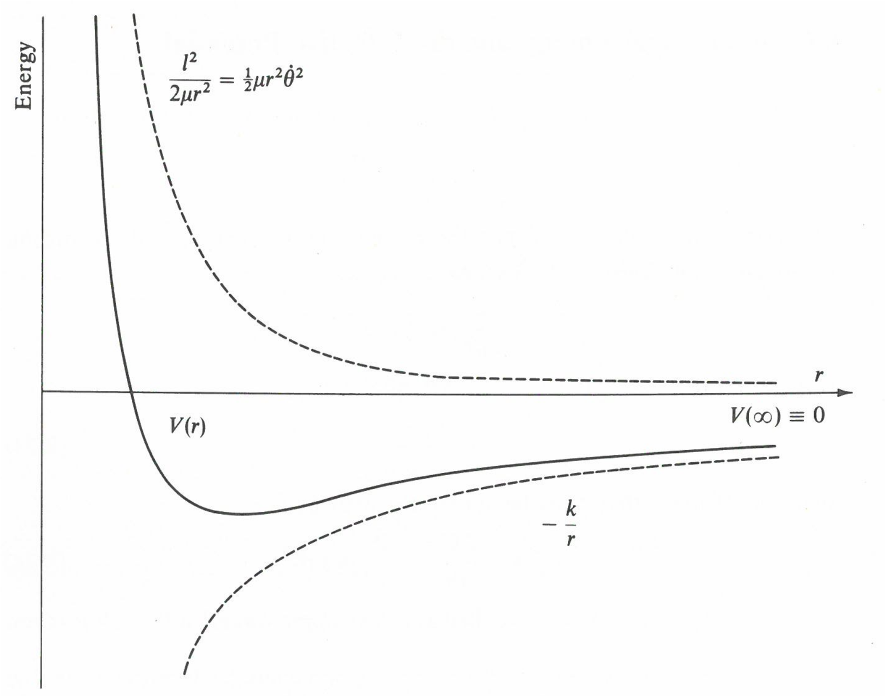
- We now continue to follow the traditional approach and consider orbits in a central field. Recall:
- $\dot{r} = \frac{dr}{dt} = \sqrt{\frac{2}{\mu}(E-U) - \frac{l^2}{\mu r^2}}$
- Thus $\dot{r}$ will be zero at the "turnaround radii", r_{min} and $r_{max.}$ This will occur when:
- $E - U(r) - \frac{l^2}{2\mu r^2} = 0$. Note that for certain values of *E, U(r)* and *l* only a single value of *r* is allowed (single root) and the orbit is then circular.

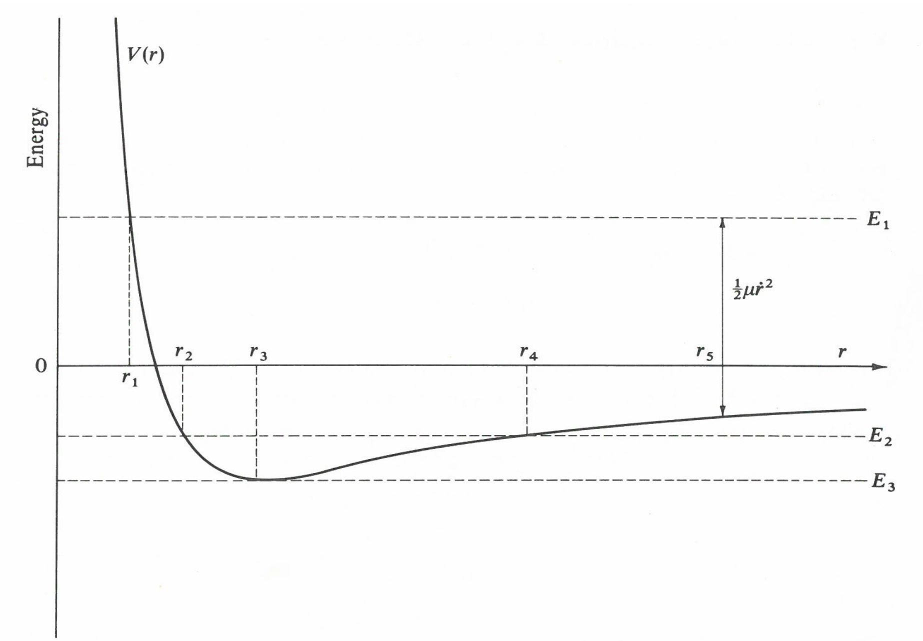


- **Centrifugal Energy and the Effective Potential**
- Recall that:
- $\frac{l^2}{2\mu r^2} = \frac{1}{2}\mu r^2\dot{\theta}$
- If we interpret this as a "potential energy" or an effective potential:
- $U_c = \frac{l^2}{2\mu r^2}$ then the corresponding force would be $F_c = -\frac{\partial U_c}{\partial r} = \frac{l^2}{\mu r^2} = \mu r\dot{\theta}^2$
- or the centrifugal force. We next introduce the idea of an effective potential:
- $V(r) = U(r) + \frac{l^2}{2\mu r^2}$
- For the gravitational potential $U(r) = -\frac{k}{r}$ we then have:
- $V(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$
- Thus, we see that if the total energy is negative the particle is bound within the
- "turning points" and if the total energy is positive it is unbound.

Effective Potential Defines Orbit

- **Orbits Are Not Ellipses Unless the Potential is ~ 1/R**
- **For a Given Angular Momentum An Orbit is Limited by the "Turning Points" (R_{min} < R < R_{max})**





Galactic Dynamics I: Disks

- **Consider the properties of stellar orbits in the disk of the Milky Way**
  - **Orbits are approximately circular but not precisely so let's see what information is available**
  - **Jan Oort parameterized stellar orbits in the following way:**

Let the radial and tangential velocites of a star be:

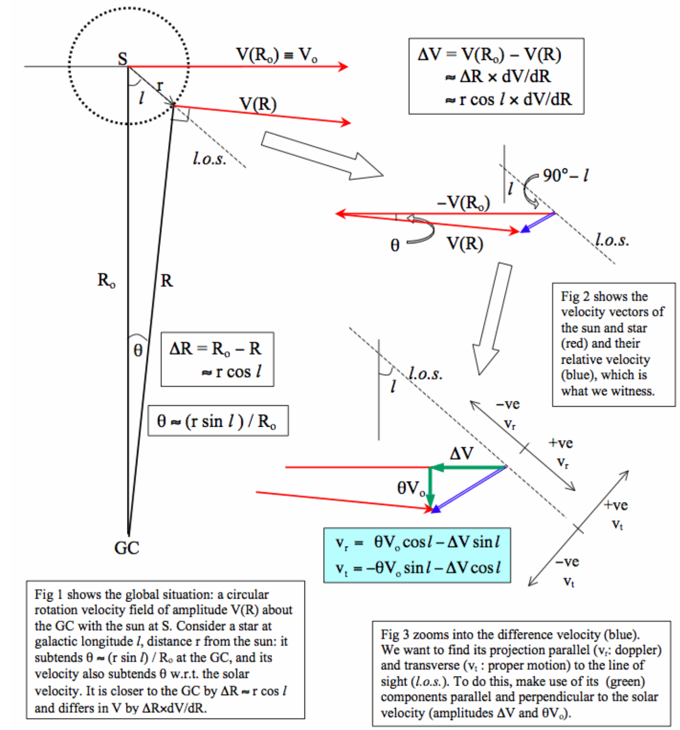
$$V_r = Ar\sin 2l$$

$$V_t = Br + Ar\cos 2l$$

where A and B are the Oort constants given by:

$$A = \frac{1}{2}\left(\frac{V_c}{R} - \frac{dV_c}{dR}\right)_{R_0} = -\frac{1}{2}R\left(\frac{d\Omega}{dR}\right)_{R_0}$$

$$B = -\frac{1}{2}\left(\frac{V_c}{R} + \frac{dV_c}{dR}\right)_{R_0} = -\frac{1}{2}R\left(\frac{d\Omega}{dR} + \frac{2\Omega}{R}\right)_{R_0}$$



- **A is a measure of the local radial gradient of the circular velocity, the shear**
- **B is a measure of the local vorticity (curl)**
- **Hipparcos yields: A = 14.8 +- 0.8 km/s/kpc, B = -12.4 +- 0.6 km/s/kpc**
- **Combining them gives some insight into local stellar orbits:**

$$A + B = \left(\frac{dV_c}{dR}\right)_{R_0} = +2.4 \text{ km/s/kpc} \quad \text{(flat rotation curve)}$$

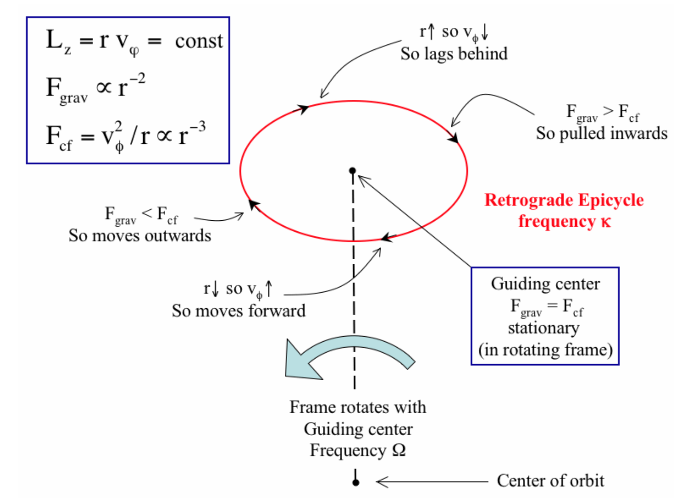
$$A - B = \left(\frac{V_c}{R}\right)_{R_o} = \Omega(R_0) = 27.2 \text{ km/s/kpc } (P = 2\pi / \Omega(R_0) = 230 \text{ Myr}$$

When combined with a measure of R_0 = 8kpc:

$V_c(R_0)$ = 218 (R_0/8 kpc) km/s

Galactic Dynamics I: Disks

- **Consider small perturbations from circular orbits**
  - **The velocity will be almost unchanged as star is perturbed radially but the small change is significant**
  - **Result is an elliptical orbit with a > R**
  - **Angular momentum must be conserved:**
    - **As r increases, V must decrease and vice versa**
  - **Stars perturbed initially outward will fall behind those on circular orbit**
    - **$F_{grav} > F_{cent}$ so stars moves back in**
  - **Stars perturbed initially inward will lead those in circular orbit**
    - **$F_{grav} < F_{cent}$ so star moves back out**
  - **The cycle repeats and so elliptical orbit can be modeled as an epicycle centered on the guiding (circular) orbit.**
  - **We define and angular velocity for the epicycle κ and it is retrograde.**
  - **For a Keplerian potential (orbit about point mass) we have:**
    - **$\Omega_g = \kappa_g$ and so the orbit is a closed ellipse**
  - **This is not true in general and so the orbits are not closed**
    - **Unless we consider a rotating frame with**
    - **$\Omega = \Omega_g - 1/2\ \kappa_g$ then orbits are closed ellipses centered on galaxy**
    - **If we have a phase shift of these orbits with radius we see a spiral-like pattern similar to that seen at the right.**
  - **Now we have a dynamical method for producing spiral arms**



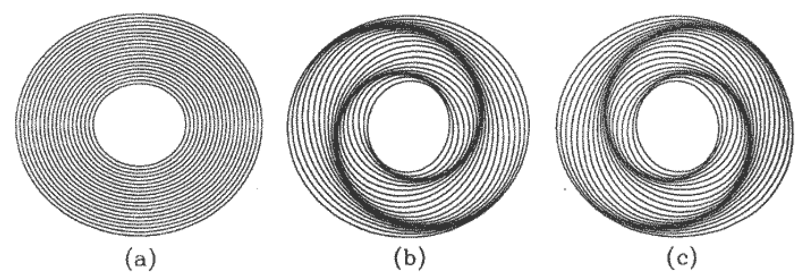


Figure 6-11. Arrangement of closed orbits in a galaxy with $\Omega - \frac{1}{2}\kappa$ independent of radius, to create bars and spiral patterns (after Kalnajs 1973).

Galactic Dynamics I: Disks

- **Dynamical basis for epicyclic motion of disk stars**
- **Consider the potential of a flattened, axisymmetric disk: Φ(R,z). Since angular momentum is conserved (no azimuthal forces):**

$\ddot{r} = -\nabla\Phi(R,z)$ and $L_z = R^2\dot{\phi} = const.$

For cylindrical coordinates (R, ϕ, z):

$$\ddot{R} - R\dot{\phi}^2 = -\frac{\partial\Phi}{\partial R} \text{ and } \ddot{z} = -\frac{\partial\Phi}{\partial z} \text{ with } \frac{d}{dt}(L_z) = 0$$

- **Note the centrifugal acceleration term.**
- **Consider z motions about the plane:**

$\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} = 0$ since the disk is continuous and symmetric about z = 0

If we expand the z-force for small z (linear terms):

$$\ddot{z} = -\left(\frac{\partial\Phi}{\partial z}\right)_{z=0} - z\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} = -z\left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0} = -\nu^2 z \text{ where } \nu^2 = \left(\frac{\partial^2\Phi}{\partial z^2}\right)_{z=0}$$

This is the equation of motion for symple harmonic motion with frequency ν and the solution is:

$z(t) = Z\cos(\nu t + \psi_0)$

For the Milky Way near the sun, $\nu^2 = 4\pi G\rho_0$ or $\nu \approx 0.096\text{Myr}^{-1}$

So the vertical oscillation period $(2\pi/\nu) \approx$ 6.5 x 10^7 yr $\approx$ 1/3 Ω

Galactic Dynamics I: Disks

- **Now consider the radial motions about the circular guiding (reference) orbit:**

$$\left(\frac{\partial\Phi}{\partial R}\right)_{R_g} = \frac{V_c^2}{R_g} = R_g\Omega_g^2$$

For non-circular orbits the equation of motion is:

$$\ddot{R} = R\dot{\phi}^2 - \frac{\partial\Phi}{\partial R}$$

We can also write this interms of the angular momentum:

$$\ddot{R} = -\frac{\partial\Phi_{eff}}{\partial R} \quad \text{where} \quad \Phi_{eff} = \Phi(R,z) + \frac{L_z^2}{2R^2} \quad \text{since } L_z = R^2\dot{\phi}$$

We can plot Φ_{eff} to ilustrate the sharp rise at small r and the slow rise at large R. The minimum occurs at the guiding orbit:

$$\left(\frac{\partial\Phi_{eff}}{\partial R}\right)_{R_g} = 0 = \left(\frac{\partial\Phi}{\partial R}\right)_{R_g} - R_g\dot{\phi}_g^2 = \left(\frac{\partial\Phi}{\partial R}\right)_{R_g} - \frac{V_c^2}{R_g}$$

For some small perturbation x consider the potential at $R = R + x$:

$$\ddot{R} = \ddot{x} = -\left(\frac{\partial\Phi_{eff}}{\partial R}\right)_{R_g} - x\left(\frac{\partial^2\Phi_{eff}}{\partial R^2}\right)_{R_g} = -x\left(\frac{\partial^2\Phi_{eff}}{\partial R^2}\right)_{R_g} = -\kappa^2 x$$

Once again this is the equation of simple harmonic motion about R_g:

$$x(t) = X\cos(\kappa t + \phi_0) \quad \text{where } \kappa^2 = \left(\frac{\partial^2\Phi_{eff}}{\partial R^2}\right)_{R_g} = \left(\frac{\partial}{\partial R}\left(\frac{\partial\phi}{\partial R}\right)\right)_{R_g} + \frac{3L_z^2}{R_g^4} \quad \text{or:}$$

$$\kappa^2 = \left(R\frac{d\Omega^2}{dR} + 4\Omega^2\right)_{R_g}$$

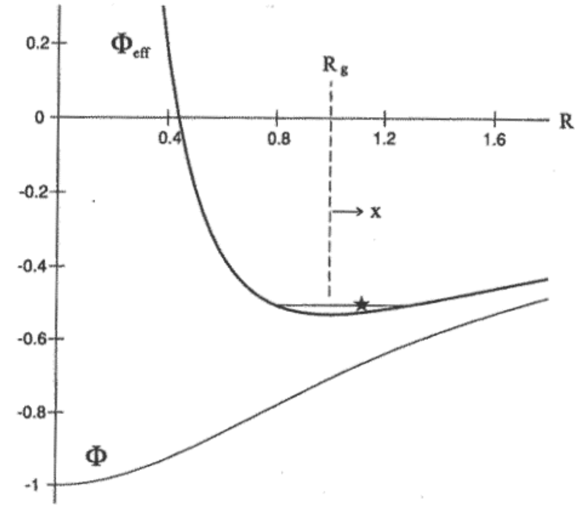


Figure 3.7 Effective potential Φ_{eff} (upper curve) for a star with angular momentum $L_z = 0.595$, orbiting in a Plummer potential Φ_P (lower curve). The scale length $a_P = 1$; L_z is in units of $\sqrt{G\mathcal{M}a_P}$; units for Φ and Φ_{eff} are $G\mathcal{M}/a_P$. The vertical dashed line marks the guiding center R_g; the star oscillates about R_g between inner and outer limiting radii.

Galactic Dynamics I: Disks

- **Now consider the azimuthal motions about the circular guiding (reference) orbit:**

Since:

$L_z = R^2\Omega_g =$ const., changes in R yield changes in Ω:

$$\Omega = \dot{\phi} = \frac{L_z}{R^2} = \frac{L_z}{(R_g + x)^2} \approx \frac{L_z}{R_g^2}\left(1 - \frac{2x}{R_g}\right) = \Omega_g\left(1 - \frac{2x}{R_g}\right)$$

Integrating yields:

$$\phi(t) = \Omega_g t - \left(\frac{2\Omega_g X}{\kappa R_g}\right)\sin(\kappa t + \phi_0)$$

Thus, ϕ follows the guiding orbit with small amplitude SHM suprposed. If we let y be the azimuthal perturbations:

$$y(t) = -\frac{2\Omega_g}{\kappa} X \sin(\kappa t + \phi_0)$$ and so the frequency κ is that same as in x (radial direction) but out of phase by 90^o. Together:

$$x(t) = X\cos(\kappa t)$$

$$y(t) = -\frac{2\Omega_g}{\kappa} X\cos(\kappa t) \quad \text{assuming } \phi_0 = 0$$

Some properties are:

· elliptical epicycle with radial/azimuthal = $\kappa/2\Omega$

· retrograde epicyclic motion

For a Keplerian potential: $\kappa = \Omega$ (closed ellipses)

For flat rotation curve: $\kappa = \sqrt{2}\Omega$

For solid body rotation: $\kappa = 2\Omega$ (closed oval orbits)

Near the Sun we predict:

$\kappa_0^2 = -4B(A - B) = -4B\Omega_0$ where $\kappa_0 = 37$ km/s/kpc and

$\Omega_0 = A - B = 27$ km/s/kpc

This corresponds to $\kappa_0 / \Omega_0 \approx 1.3$ (stars make 1.3 cycles per orbit) and since $\kappa_0 / 2\Omega_0 \approx 0.7$ then epicycles have radial/azimuthal ≈ 0.7

The observed velocity ellipsoid at $R_g = R_0$ is:

$\sigma_R / \sigma_\phi = \kappa_0 / 2\Omega_0 \approx 0.7$ in good agreement, but:

$\sigma_R / \sigma_\phi = 2\Omega_0 / \kappa_0 \approx 1.5$ because there are more stars at smaller R

This results in about 1 kpc excursions in R. Similarly, for z:

$\sigma_z \approx 30$ km/s with $\nu \approx 0.096$ Myr^{-1} with excursions of about 300 pc.