- **Chapter 15: Stellar Interiors**
  - **Physical Laws of Stellar Structure**
    - **Hydrostatic Equilibrium**
    - **Equation of State**
    - **Energy Transport (Radiative Transfer)**
    - **Energy Sources**
  - **Theoretical Stellar Models**
    - **Physical Principles**
    - **Computing Stellar Models**
  - **Synthesis of Heavy Elements in Stars**

Chapter 15 Homework

Chapter 15: #1, 2, 4, 10

(Due Tues. March 28)

Equations of Stellar Structure

Stellar physics can be specified by a set of 4 1st-order differential equations and an equation of state for the gas.

Hydrostatic Equilibrium:
$$dP/dr = -[GM(r)\rho(r)]/r^2$$

Mass Conservation:
$$dM/dr = 4\pi r^2\rho(r)$$

Energy Generation:
$$dL/dr = 4\pi r^2\rho(r)\,\varepsilon(r) \qquad [\varepsilon(r) \text{ is the emissivity at radius } r]$$

Opacity and Energy Transport:
$$dT/dr = -[3\kappa(r)\rho(r)]/64\pi\sigma r^2 T^3(r)]L(r)$$

Equation of State:
$$P(r) = [\rho(r)kT(r)]/[\mu(r)m_H] \quad \text{(ideal gas law with } \mu(r) \text{ as the mean mole. weight)}$$
And $P\gamma(r) = [4\sigma/3c]T^4(r)$ **(radiation pressure)**

These equations are solved simultaneously and numerically with boundary conditions such that $M(R) = M$, $L(R) = L$, $T(R) = T_{eff}$, $P(R) = 0$, $\rho(R) = 0$.

See pages 350-352 for estimating central pressure and temp. in Sun.

Russell-Voit Theorm

- **The physical properties of a star (luminosity, radius, and temperature) are determined by its mass and chemical composition.**

- **Stars evolve because they change their chemical composition.**

  - **Nuclear fusion changes the composition of the core.**

Radiative Transfer - I

- **Intensity vs. Flux**
    - **Intensity: the energy emitted by a region/surface per unit area. Per unit time, per unit frequency (or per unit wavelength), per unit solid angle (fraction of the sky).**
    - **Solid angle is the 3-d equivalent to the arc length:**
    - $\Delta\Omega = \Delta$**area**$/r^2$ **where is the fraction of the sky x 4π subtended by an area Δarea at a distance r from the origin.**
    - **Note that the entire sky subtends 4π steradians and an angle of 1 radian corresponds to a solid angle of 1 steradian.**
    - **Consider now a particular object which emits some energy spectrum $I(\lambda)$. If we integrate over the entire surface and over the spectrum (λ, or ν), we can compute the _luminosity_. In the case of a spherical object with uniform $I(\lambda)$:**

    $$L = 4\pi r^2 \text{ Integral } [I(\lambda)] \, d\lambda = 4\pi r^2 \, \sigma T^4 \text{ if Blackbody}$$

    - **We can now define the flux (F) as the energy per unit time and per unit wavelength which passes through a unit area. If an objects emits uniformly in all directions then:**

    $$F = L/4\pi r^2 \qquad \text{(the inverse square law)}$$

    - **Absorption of Light**

Radiative Transfer - II

Consider a source emitting a beam with spectrum I(λ), which encounters an absorbing medium. In this simple case:

$dI/I = - \chi \, dI$ where χ is the extinction coefficient and describes the absorbing properties density, etc. and dl represents the path length through the medium. Integrating we have:

$$I = I_0 \, e^{-\tau}$$ that is the intensity declines exponentially.

The more general form of this differential equation allows for emission within the medium, not just absorption. This is called the <u>transfer equation</u>:

$dI_\lambda/dl = \eta_\lambda - \chi_\lambda I_\lambda$ where   is called the emisssion coefficient and represents the energy emitted by the gas per unit volume, per unit steradian, per unit wavelength, per second. It is customary to consider an observer looking into the source. Thus we introduce a minus sign since now I = - I. In addition, we note that any emitting region is seen in projection ($\mu = \cos \theta$) from an arbitrary location. So if we divide through by χ_λ we have:

$\mu dI_\lambda/dl = I_\lambda - S_\lambda$ where S_λ is the ratio of emission to absorption for the gas and is known as the source function.

• See the text for some simple solutions.

Boltzman & Saha Equation

- **Thermal Equilibrium**
  - **Recall that the Boltzman distribution describes the number of atoms or molecules as a function of energy.**
    - **On average:**
      - $<mv^2/2> = 3kT/2$
  - **Collisions between these can change their *internal* energy**
    - **Electrons can be excited into upper levels**
    - **Electrons can be collisionally de-excited as well.**

- **Boltzman Equation**
  - **Describes the number of atoms with electrons in a given state according to the energy difference between the excited levels:**

    $$N_B/N_A = (g_B/g_A)e^{-(E_A - E_B)/kT}$$

    where the g's are the statistical weights and contain the multiplicity of the levels (statistical weights) and the transition probabilities (quantum mechanics).

- **Saha Equation**
  - **Saha showed that a similar relation exists for atoms with differing ionization states:**

    $$N_+/N_0 = [(2\pi mkT/h^2)^{3/2} /N_e]B_+/B^0 \, e^{-\chi//kT}$$

    Where N_e is the number density of electrons.

Modes of Energy Transport

- **Recall that hydrostatic equilibrium requires that the central temperature of the Sun is ~ 15 million degrees. With a surface temperature of about 6000 degrees the heat flows outward until it can escape, i.e., the optical depth of the gas < 1.**

- **Three modes of energy transport:**

  - **Conduction: Atomic collisions transport energy (heat) from hotter regions to cooler. Works best in dense gas, liquids, or solids where the atoms are densely packed and strongly interacting. Not important for most stars since they are gas (White Dwarfs an exception)**

  - **Convection: Energy is transported via the motion of matter. In a liquid or gas this takes the form of circulation currents. Hotter gas expands and rises. When temp. gradient (dT/dr) is large enough gas becomes unstable and boils (convection).**

  - **Radiative Transport: If dT/dr is low and temperatures high photons can transport a significant amount of energy. See the derivation of the energy transport equation in text (pg 353-359).**

Modes of Energy Transport-II

- Convection: Hot gas expands and density drops so gas can rise. If a perturbed (hot) blob has lower density than surroundings (recall density gradient) it will rise. If no heat is lost (adiabatic) we can derive a simple stability criteria (eq. 15.40 and 15.43).

If the temperature gradient is too high for radiation to carry away the energy (opacity too high) gas begins to boil (convect).

$$\frac{1}{\gamma P}\frac{dP}{dr} > \frac{1}{\rho}\frac{d\rho}{dr}$$ where γ is the adiabatic index (5/3 for ionized gas) and PV^{γ} = const.

This criteria can also be expressed in terms of the termperature gradient:

$$\left(1-\frac{1}{\gamma}\right)\frac{T}{P}\frac{dP}{dr} = \frac{dT}{dr}$$ (convective transport) while radiation can transport:

$$\frac{dT}{dr} = -\frac{3\rho(r)\kappa(r)L(r)}{64\pi\sigma T^3(r)r^2}$$ (radiative transport)
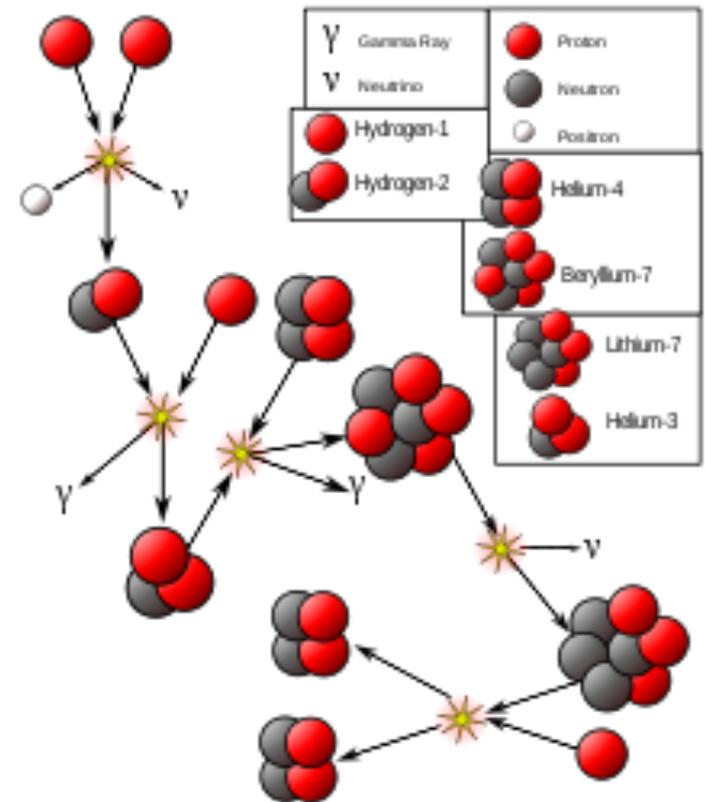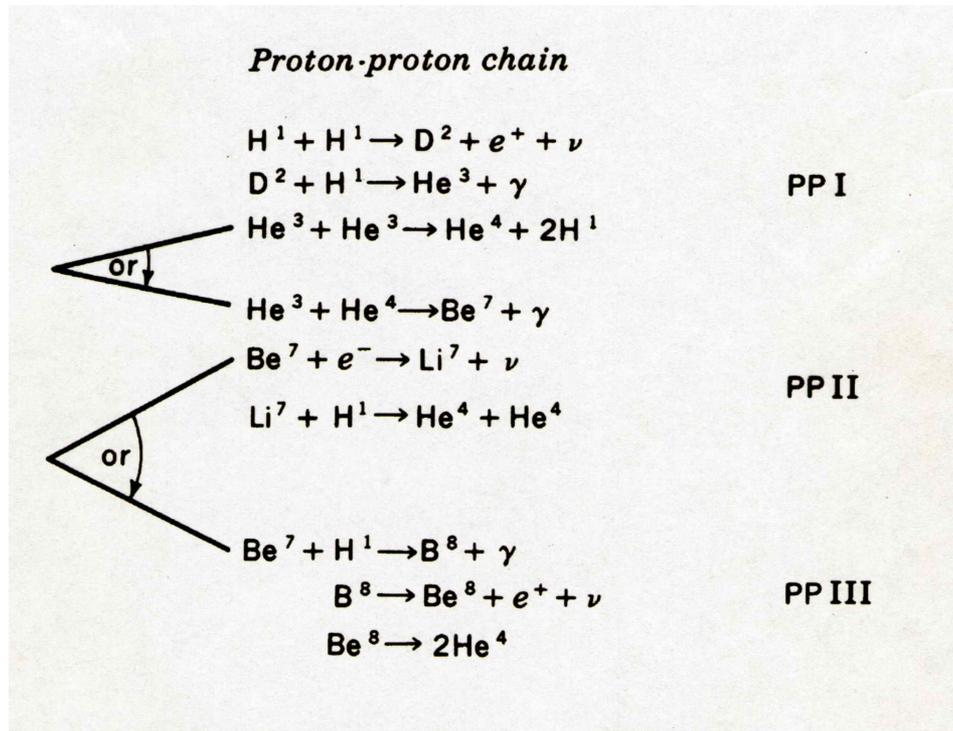
The inner 70% of the Sun is radiative, the outer 30% is convective.

Energy Generation in Stars

- Gravitational Contraction
  - Change in potential energy is converted into heat
    - This is the process by which a star begins to radiate energy and the central temperature rises (proto-stellar phase).
- Thermonuclear Reactions
  - Deuterium burning
    - Deuterium ignites first but is rare and can't provide much fuel
  - Proton-proton Chain
    - Main mechanism in medium – low mass stars
  - CNO Cycle
    - C, N, and O act as catalysts to burn Hydrogen to Helium
  - Triple-alpha Process
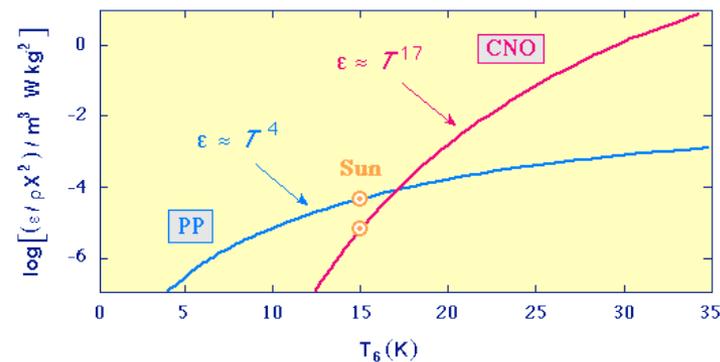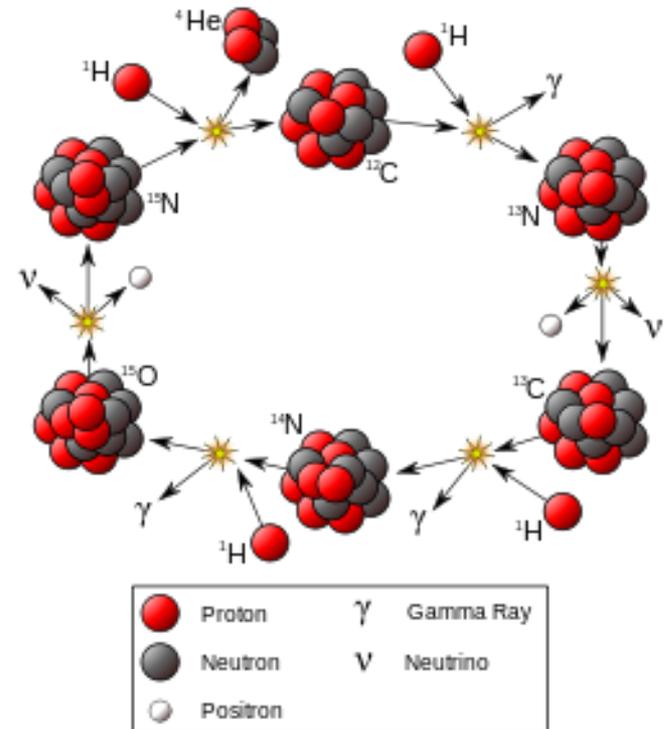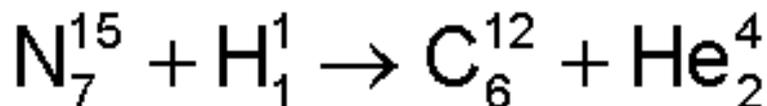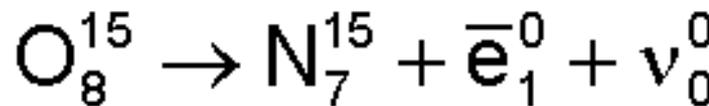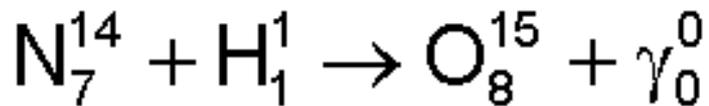   - At high temps. $3He^4$ can fuse to form C^{12}

Fusion Reactions

- **P-P Chain**

Proton·proton chain

$$H^1 + H^1 \rightarrow D^2 + e^+ + \nu$$
$$D^2 + H^1 \rightarrow He^3 + \gamma \qquad\qquad \text{PP I}$$
$$He^3 + He^3 \rightarrow He^4 + 2H^1$$

or

$$He^3 + He^4 \rightarrow Be^7 + \gamma$$
$$Be^7 + e^- \rightarrow Li^7 + \nu \qquad\qquad \text{PP II}$$
$$Li^7 + H^1 \rightarrow He^4 + He^4$$

or

$$Be^7 + H^1 \rightarrow B^8 + \gamma$$
$$B^8 \rightarrow Be^8 + e^+ + \nu \qquad\qquad \text{PP III}$$
$$Be^8 \rightarrow 2He^4$$

γ Gamma Ray	Proton
ν Neutrino	Neutron
Hydrogen-1	Positron
Hydrogen-2	Helium-4
	Beryllium-7
	Lithium-7
	Helium-3

Fusion Reactions

- **CNO Cycle:**

$$C_6^{12} + H_1^1 \rightarrow N_7^{13} + \gamma_0^0$$

$$N_7^{13} \rightarrow C_6^{13} + \overline{e}_1^0 + \nu_0^0$$

$$C_6^{13} + H_1^1 \rightarrow N_7^{14} + \gamma_0^0$$

$$N_7^{14} + H_1^1 \rightarrow O_8^{15} + \gamma_0^0$$

$$O_8^{15} \rightarrow N_7^{15} + \overline{e}_1^0 + \nu_0^0$$

$$N_7^{15} + H_1^1 \rightarrow C_6^{12} + He_2^4$$

● Proton	γ	Gamma Ray
● Neutron	ν	Neutrino
○ Positron		

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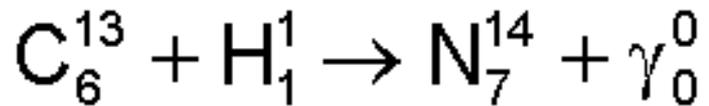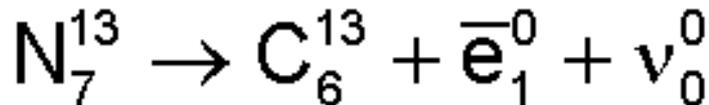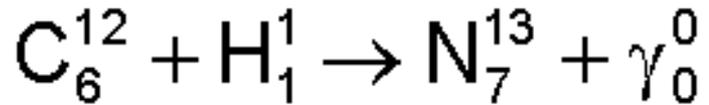
Modeling Stellar Interiors

- **Simultaneously solving the stellar structure equations requires a numerical solution**
  - **First re-write them as finite difference equations**
  - **Second construct a model as a series of thin radial shells and solve the equations for each shell allowing for energy transport.**
  - **Starting at the boundary (r = R*) compute the physical properties and compute derivatives.**
  - **Apply this to the next shell inward and compute it's physical properties. Repeat until the center of the star is reached. Compute the nuclear reaction rates in the appropriate shells and the total emissivity for each shell (blackbody + any fusion).**
  - **Choose a time step and compute the change in the mean molecular weight as fusion products accumulate. Recompute the equation of state and its effects on the physical properties and recompute the entire model.**
  - **Repeat as necessary to model the star's evolution with time.**
  - **Compute the observable properties vs. age: Luminosity and Temperature. Result is an evolutionary track in the H-R diagram.**

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